
Exercise sheet 7

31/10/2025

Exercise 1.

Let $(B_t)_{t \geq 0}$ be a standard Brownian motion.

- (a) For all $n \geq 1$, define $\tau_n = \inf \left\{ t > 0 : \int_0^t e^{2B_s^4} ds \geq n \right\}$. Show that for all $t > 0$, $\mathbb{P}\{\tau_n \leq t\} > 0$.
- (b) Explain how to determine a random variable $\int_0^t e^{B_s^4} dB_s$ from simple predictable process.

Exercise 2.

- (a) Show that if \mathcal{F} is a sigma-algebra and if $(X_n)_{n \in \mathbb{N}}$ is a sequence of random variables such that $X_n \xrightarrow[n \rightarrow \infty]{} X$ a.s. and $|X_n| \leq Y \in L^1$, then

$$\mathbb{E}(X_n | \mathcal{F}) \xrightarrow[n \rightarrow \infty]{} \mathbb{E}(X | \mathcal{F}), \text{ in } L^1 \text{ and a.s.}$$

- (b) Let $(X_t, t \in \mathbb{R}_+)$ be a continuous local martingale and $T > 0$ such that $E \left(\sup_{0 \leq s \leq T} |X_s| \right) < +\infty$. Show that $(X_t, t \in [0, T])$ is a martingale.

Exercise 3.

Let $M = (M_t, t \in \mathbb{R}_+)$ be a bounded continuous martingale. Define

$$S_t^n = \sum_{i=1}^n \left(M_{t_i^{(n)}} - M_{t_{i-1}^{(n)}} \right)^2$$

where $0 = t_0^{(n)} < t_1^{(n)} < \dots < t_n^{(n)} = t$ and $\lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} (t_i^{(n)} - t_{i-1}^{(n)}) = 0$. Show that

$$\lim_{n \rightarrow \infty} S_t^n = \langle M \rangle_t, \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P}).$$