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**Problem Set 4**

6.10.2025

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*Part 1*

**Exercise 1.**

Let  $(B_t, t \in \mathbb{R}_+)$  be a standard Brownian motion. For  $a, b > 0$ , define

$$T_{-a,b} = \inf\{t \geq 0 : B_t > b \text{ ou } B_t < -a\}.$$

Here we admit that  $T_{-a,b}$  is a stopping time.

- (a) Show that  $B_{T_{-a,b}} \in \{-a, b\}$ , a.s.
- (b) Show that  $\mathbb{E}(B_{T_{-a,b}}) = 0$  and deduce that  $\mathbb{P}\{B_{T_{-a,b}} = b\} = \frac{a}{a+b}$ .
- (c) Show that  $\mathbb{E}(B_{T_{-a,b}}^2) = \mathbb{E}(T_{-a,b})$  and deduce that  $\mathbb{E}(T_{-a,b}) = ab$ .

*Part 2*

**Exercise 2.**

Let  $p > 0$ . Given  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  a continuous function, and define

$$V_p(t) = \limsup_{n \rightarrow +\infty} \sum_{j=1}^{2^n} |f(jt2^{-n}) - f((j-1)t2^{-n})|^p.$$

Let  $q > p$ . Show that if  $V_p(t) < +\infty$ , then  $V_q(t) = 0$ . We note that the contra-posed statement is : If  $V_q(t) > 0$ , then  $V_p(t) = +\infty$ .

**Exercise 3. (Integral of a step function with respect to a function with bounded variation)**

- (a) Let  $g : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that  $g = g_1 - g_2$ , where  $g_1$  and  $g_2$  are increasing functions. Show that by a direct calculation,  $g$  is a function with (locally) bounded variation.
- (b) Let  $g$  be a continuous function with bounded variation and let  $0 \leq t_1 < t_2 \leq t$ . Show that

$$\int_0^t \mathbf{1}_{]t_1, t_2]}(s) dg(s) = g(t_2) - g(t_1).$$

- (c) Let  $g$  be a continuous function with bounded variation. Let  $0 = t_0 < t_1 < \cdots < t_n = t$  and  $f$  be a step function defined by

$$f(s) = \sum_{i=1}^n a_i \mathbf{1}_{]t_{i-1}, t_i]}(s), \text{ where } a_i \in \mathbb{R}, i = 1, \dots, n.$$

Deduce from point (b) that

$$\int_0^t f(s) dg(s) = \sum_{i=1}^n a_i (g(t_i) - g(t_{i-1})).$$