
Exercise sheet 3

26/09/2025

Exercise 1. (Properties of stopping time)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t, t \in \mathbb{R}_+)$ be a filtration. Show following properties.

- (a) If T is a stopping time, then for all $t \geq 0$, events $\{T < t\}$, $\{T \geq t\}$, $\{T > t\}$ are belong to \mathcal{F}_t .
- (b) If S and T are two stopping times, then for $S \vee T$ and $S \wedge T$ are stopping times.
- (c) If $(T_n, n \in \mathbb{N})$ is a sequence of stopping time, then $\sup_{n \in \mathbb{N}} T_n$ is a stopping time.
- (d) If a filtration is right continuous (i.e., for all $t \geq 0$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$), and if $(T_n, n \in \mathbb{N})$ is a sequence of stopping times, then $\inf_{n \in \mathbb{N}} T_n$ is a stopping time.

Exercise 2.

Let $(B_t, t \in \mathbb{R}_+)$ standard Brownian motion and $(\mathcal{F}_t^B, t \in \mathbb{R}_+)$ is the natural filtration. Show that the processes $(B_t^2 - t, t \in \mathbb{R}_+)$ and $(\exp\{B_t - \frac{t}{2}\}, t \in \mathbb{R}_+)$ are martingales with respect to (\mathcal{F}_t^B) .

Exercise 3. (Doob Martingale)

Let X be an integrable random variable and $(\mathcal{F}_t, t \in \mathbb{R}_+)$ is a filtration. Define $M_t = \mathbb{E}(X|\mathcal{F}_t)$.

- (a) Show that $(M_t, t \in \mathbb{R}_+)$ is a martingale with respect to (\mathcal{F}_t) .
- (b) Let $s \in \mathbb{R}_+$. What can we say about the processes (M_t) if X is \mathcal{F}_s -mesurable?

Exercise 4.

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $(\mathcal{F}_t, t \in [0, 1])$, and let H^2 be the set of continuous martingales $M = (M_t, t \in [0, 1])$, equipped with norm $\|M\|^2 = \mathbb{E}(M_1^2)$. Show that H^2 is a Hilbert space.