Exercise sheet 1 05/09/2025

Exercise 1.

Let $(B_t,\, t \geqslant 0)$ be a standard Brownian motion and let $t_0 = 0 < t_1 < \cdots < t_n$. Define $X = (B_{t_1}, B_{t_2}, \ldots, B_{t_n})^t$ and $Y = (B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})^t$. Let A be the $n \times n$ diagonal matrix given by $\mathrm{Diag}(\sqrt{t_1}, \sqrt{t_2 - t_1}, \ldots, \sqrt{t_n - t_{n-1}})$ and T the following $n \times n$ matrix

$$T = \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & \ddots & \ddots & \\ 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Define $\Sigma = TA$.

(a) Verify that X follows a multivariate normal distribution $\mathcal{N}_n(\vec{0}, \Sigma\Sigma^t)$. That is, the joint density function of X is given by

$$f_X(x) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \frac{1}{\det(\Sigma)} \exp\left\{-\frac{1}{2} x^t \left(\Sigma\Sigma^t\right)^{-1} x\right\}, \text{ for all } x \in \mathbb{R}^n.$$

(b) Show that for $\theta \in \mathbb{R}^n$,

$$\mathbb{E}\left(\exp(i\theta \cdot Y)\right) = \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \theta_j^2 (t_j - t_{j-1})\right)$$

and deduce that

$$\mathbb{E}\left(\exp(i\theta \cdot X)\right) = \exp\left(-\frac{1}{2}\, \theta^t (\Sigma\Sigma^t)\theta\right).$$

Exercise 2.

Let $(X_n,\, n \geqslant 0)$ be an i.i.d sequence of $\mathcal{N}(0,1)$ random variables. For $l, p \in \mathbb{N}$ with $l < p$, define

$$T_{l,p} = \sup_{t \in [0,\pi]} \left| \sum_{n=l}^{p-1} X_n \frac{\sin(nt)}{n} \right|.$$

Show that

$$\mathbb{E}\left((T_{l,p})^2\right) \leqslant \frac{p-l}{l^2} + 2\frac{(p-l)^{\frac{3}{2}}}{l^2}.$$

Exercise 3.

Let $p(t, x, \cdot)$ be the conditional density of B_t given that $B_0 = x$. Show that

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}.$$