Exercise Series 13 12/12/2025

Exercise 1. (Extension of the Feynman-Kac Formula)

Let $q, f, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$ be bounded Borel functions. Assume that the equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(t,x) + \mu(x)\frac{\partial u}{\partial x}(t,x) + q(x)u(t,x), \quad t \in \mathbb{R}_+,\, x \in \mathbb{R},$$

with the initial condition $u(0,x) = f(x)$, $x \in \mathbb{R}$, admits a unique bounded solution, such that $\frac{\partial u}{\partial x}$ is bounded.

Assume that $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function and therefore there exists a constant $C \in \mathbb{R}_+$ such that for all $x \in \mathbb{R}$,

$$|\mu(x)| \leqslant C(1 + |x|).$$

Show that

$$u(t,x) = \mathbb{E}\left(f\left(X_t^{(x)}\right) \exp\left(\int_0^t q\left(X_s^{(x)}\right)\, ds \right)\right),$$

where $(X_t^{(x)})$ is the unique solution of the stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t, \quad X_0 = x.$$

Exercise 2.

Let $T > 0$. Let (B_t) be a standard Brownian motion. For each of the random variables X below, find a process $H \in \mathcal{H}_T(B)$ such that

$$X = \mathbb{E}(X) + \int_0^T H_s\, dB_s. \qquad (*)$$

(a) $X = B_T$

(b) $X = \int_0^T B_s\, ds$

(c) $X = B_T^2$

(d) $X = B_T^3$.