Problem Set 12 7.12.2025

Exercise 1.

Let $(B_t, t \in \mathbb{R}_+)$ be a standard Brownian motion in $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Find the density function of $B_{\frac{2}{3}}$ under the probability $\mathbb{Q}$ defined by $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}\left(X\left(e^{B_1-\frac{1}{2}}\right)\right)$, where X is a $\mathcal{F}_1$-mesurable r.v.

(b) Find a probability measure under which $(W_t, t \in [0,2])$ is a Brownian motion, where $W_t = B_t + t - t^3$, $t \in [0,2]$.

Exercise 2. (Weak solutions of stochastic differential equations)

Let $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel function. We consider following equation :

$$dX_t = f(t, X_t)\,dt + dB_t. \tag{1}$$

A *weak solution with initial condition* $x_0 \in \mathbb{R}$ of this equation is a quadruplet $((X_t), (B_t), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t))$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(\mathcal{F}_t)$ is filtration on $(\Omega, \mathcal{F})$, (B_t) is a standard Brownian motion adapted in $(\mathcal{F}_t)$ and (X_t) is a continuos process adapted in $(\mathcal{F}_t)$, such as $\mathbb{P}\left\{\int_0^t |f(s, X_s)|\,ds < +\infty\right\} = 1$ and

$$X_t = x_0 + \int_0^t f(s, X_s)\,ds + B_t, \quad t \in [0,T].$$

Suppose that there exists $K \in \mathbb{R}_+$ such that $|f(t,x)| \leqslant K$, for all $t \in [0,T]$ and $x \in \mathbb{R}$. Show that (1) has a weak solution.

Exercise 3.

Let $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ a bounded Borel function and $((X_t), (B_t), (\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t))$ be a weak solution of equation (1) in Exercise 2.

(a) Using Girsanov's theorem, show that for all $K < \infty$ and $t \in \mathbb{R}_+$, $\mathbb{P}\{X_t \geqslant K\} > 0$.

(b) In (1), let $f(t,x) = -1$. Then what can we say about $\lim\limits_{\substack{n \to +\infty \\ n \in \mathbb{N}}} X_n$?