Exercise 11 28/11/2025

Exercice 1. (Equations linéaires)

Let $A, a, \sigma : \mathbb{R}_+ \to \mathbb{R}$ be continuous bounded functions and $x_0 \in \mathbb{R}$. We consider following stochastic differential equations

$$dX_t = (A(t)X_t + a(t))\ dt + \sigma(t)\, dB_t \tag{1}$$

with initial condition $X_0 = x_0$. Let $\Phi(t)$ be the unique (continuous) solution to the differential equation (deterministic)

$$d\Phi(t) = A(t)\Phi(t)\, dt, \quad \Phi(0) = 1.$$

(a) Show that

$$\xi(t) = \Phi(t)\left(\xi(0) + \int_0^t \Phi^{-1}(s)a(s)\, ds\right)$$

the unique (continuous) solution to the differential equation (deterministic) : $\dot{\xi}(t) = A(t)\xi(t) + a(t)$.

(b) Show that

$$X_t = \Phi(t)\left(x_0 + \int_0^t \Phi^{-1}(s)a(s)\, ds + \int_0^t \Phi^{-1}(s)\sigma(s)\, dB_s\right)$$

is the solution to (1).

Exercice 2. (The Brownian bridge)

On the interval $[0, 1[$, we consider following stochastic differential equation :

$$dX_t = -\frac{X_t}{1-t}\, dt + dB_t, \quad X_0 = 0.$$

(a) Show that $X_t = (1-t)\int_0^t \frac{1}{1-s}\, dB_s$, $t \in [0, 1[$, is the solution to this equation

(b) Calculate $\mathbb{E}(X_t X_s)$ for $s, t \in [0, 1[$.

(c)We define $Y_t = B_t - tB_1$, $t \in [0, 1[$. Prove that $(Y_t,\, t \in [0, 1[)$ and $(X_t,\, t \in [0, 1[)$ have the same law.

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Exercice 3. (Characterization of Brownian motion according to Paul Lévy)

Define $(X_t, t \in \mathbb{R}_+)$be a continuous martingale and such that $X_0 = 0$ and $\mathbb{E}(X_t^2) < +\infty$, for all $t \in \mathbb{R}_+$. We suppose that $\langle X \rangle_t = t$, for all $t \in \mathbb{R}_+$. Prove that $(X_t, t \in \mathbb{R}_+)$ is a standard Brownian motion.

Exercice 4. (Characterization of martingales)

(a) Let $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be a filtration. Let $s < t$ and $A \in \mathcal{F}_s$. Prove that the random variable τ defined by

$$\tau(\omega) = \begin{cases} s & \text{if } \omega \in A, \\ t & \text{if } \omega \in A^c, \end{cases}$$

is a stopping time.

(b) Prove that an integrable continuous process $(X_t,\, t \in \mathbb{R}_+)$ adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ is a martingale if and only if, for every bounded $(\mathcal{F}_t)$-stopping time τ, $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$.