Exercise 10 21/11/2025

Exercice 1.

Let $(B_t, t \geqslant 0)$ be a standard Brownian motion.

(a) Let $(H_t, t \in [0,T])$ and $(K_t, t \in [0,T])$ be two processes in $\mathcal{H}_T(B)$. Define $X_t = \int_0^t H_s\, dB_s$ and $Y_t = \int_0^t K_s\, dB_s$. Show that for all $t \in [0,T]$,

$$X_t Y_t = \int_0^t (H_s Y_s + K_s X_s)\, dB_s + \int_0^t H_s K_s\, ds.$$

(b) Is $(X_t = t^2 B_t - 2\int_0^t s B_s\, ds,\ t \geqslant 0)$ a martingale ?

(c) Show that $(N_t = B_t^3 - 3tB_t,\ t \geqslant 0)$ is a martingale.

Exercice 2. **(Black-Scholes equation)**

Let $\mu \in \mathbb{R}$, $\sigma > 0$ and $x_0 > 0$. We consider following stochastic differential equation :

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t$$

with the initial condition $X_0 = x_0$. Show that

$$X_t = x_0 \exp\left\{\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right\}$$

is the solution.

Exercice 3. **(Langevin equation and Vasicek model)**

Let $a, b, x_0 \in \mathbb{R}$ and $\sigma > 0$. Consider following stochastic differential equation :

$$dX_t = a\,(b - X_t)\ dt + \sigma\, dB_t$$

with the initial condition $X_0 = x_0$. Show that

$$X_t = (x_0 - b)e^{-at} + b + \sigma \int_0^t e^{-a(t-s)}\, dB_s$$

is the solution.

Exercice 4. (Brownian motion on the circle)

We consider the differential system

$$\begin{cases} dX_t &= -\frac{1}{2}X_t\,dt - Y_t\,dB_t, \\ dY_t &= -\frac{1}{2}Y_t\,dt + X_t\,dB_t. \end{cases} \qquad X_0 = 1,\ Y_0 = 0$$

a) Using Itô's formula, show that $X_t^2 + Y_t^2 = 1$ a.s., for all $t \in \mathbb{R}_+$.

b) Show that $X_t = \cos(B_t)$, $Y_t = \sin(B_t)$ is the solution to this system.