1 Setup

Exactly as in class. Let $G = (V, E)$ be a finite connected graph and $V_\partial \subset V$ a non-empty boundary. Let $V^\circ = V \setminus V_\partial$ be the interior. For $x \in V$, denote by d_x the degree of x.

We will consider a probability measure on functions $f : V \to \mathbb{R}$ with $f(v) = 0$ for $v \in V_\partial$.

The aim is to show that the following are equivalent.

Theorem 1. *The following give the same probability measure:*

A. *We consider the probability law on the space of $h : V \to \mathbb{R}$ with $h(v) = 0$ at $v \in V_\partial$ given by density at point h given by*

$$\frac{1}{Z} \exp\left(-\frac{1}{2}\sum_{v\sim w}(h(v) - h(w))^2\right),$$

where by convention the sum is over unordered edges and Z is a numerical factor that normalises to give a probability measure.

B. *We consider i.i.d. standard Gaussians $g(e)$ on the directed edges of the graph, such that $g(e) = g(-e)$ if we reverse the edge. We then condition this field to sum to zero over any cycle or any boundary to boundary path and under the conditional measure consider the resulting height function defined at any vertex v by just taking any path from a boundary point and summing up $g(e)$.*

C. *We consider the zero mean Gaussian process on G with covariance given by*

$$G(v, w) := d_w^{-1}\mathbb{E}(\sum_{t=0}^{\tau} 1_{W_v(t)=w}),$$

where W_v is simple random walk starting from v and τ hitting time of the boundary.

D. *We consider the following Markov chain: pick at time $t \in \mathbb{N}$ any vertex v and resample its value as follows. We condition on the field Γ at $w \in V \setminus \{v\}$, take the average of the neighbours of v, and add an independent Gaussian of variance d_v i.e. get*

$$Z + d_x^{-1}\sum_{w\sim v}\Gamma(w),$$

where $Z \sim \mathcal{N}(0, d_x^{-1})$. We take the unique stationary measure of this chain.

Some tipps:

- The equivalence between A-D should be clear from direct computation. [1]
- To connect C and A one needs to see the relation between G and Δ. [2]
- To connect to B you should figure out the projection map and use the theory of conditioning of Gaussian fields. [3]

[1]one needs to just verify that this chain does have a unique stationary measure and that the DGFF does satisfy such sampling

[2]Discrete integration by parts could be useful, which you are able to derive.

[3]Think about taking $\nabla(\Delta^{-1}(\mathrm{div}(g)))$. Can you prove that this gives the right projection relating to the conditioning?