GAUSSIAN PROCESSES
EXERCISE SHEET 3: DISCRETE ENTROPY

Definition (Entropy)**.** *Let X be a discrete random variable with outcome space Ω. For each $\omega_x \in \Omega$, denote*

$$p(\omega_x) := \mathbb{P}(X = \omega_x).$$

The entropy of X is defined by

$$H(X) := -\mathbb{E} \log_2 p(X) = -\sum_{\omega_x \in \Omega} p(\omega_x) \, \log_2 p(\omega_x).$$

Exercise 1 (Entropy of the geometric distribution)**.** *Let X be geometrically distributed with parameter $p \in (0,1)$ on $\{1, 2, \dots\}$. That is, for $k \geq 1$,*

$$p(\omega_k) := \mathbb{P}(X = k) = p(1-p)^{k-1}.$$

Compute the entropy $H(X)$.

Exercise 2 (Basic properties of entropy)**.** *Let X, Y be discrete random variables. Prove the following:*

(1) $H(X) \geq 0$. *Moreover, if X and Y are independent then*

$$H(X,Y) = H(X) + H(Y).$$

(2) $H(X,Y) = H(X) + H(Y \mid X)$, *where the conditional entropy*

$$H(Y \mid X) := \sum_{\omega_x} p(\omega_x) \, H(Y \mid X = \omega_x).$$

(3) $H(X,Y) \leq H(X) + H(Y)$. *Moreover, equality holds if and only if X and Y are independent.*