Stochastic Simulations

Autumn Semester 2025

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Project 2

Submission deadline: 11 January 2026

Importance Sampling by Optimal Control in Option Pricing

1 Introduction and background

Importance sampling is a popular variance reduction technique used to estimate the probability of rare-events. In practice, by performing a change of measure, one would like to render such rare events more likely to occur in order to estimate more efficiently their probability.

This project investigates how to choose in an efficient manner this importance measure in the context of option pricing. We assume that the underlying asset satisfies the Stochastic Differential Equation (SDE)

$$\begin{cases} dS(t) = b(t, S(t))dt + \Sigma(t, S(t))dW(t) \quad t \in (0, T] \\ S(0) = S_0, \end{cases} \tag{1}$$

where $W(t)$ denotes a standard one-dimensional Wiener process. The price of a derivative written on (1) reads

$$Z = \mathbb{E}[\psi(S(T))], \tag{2}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}$ is the payoff function and can be approximated using Monte Carlo (MC) techniques, namely, by simulating N trajectories $S^{(1)}, ..., S^{(N)}$ of (1), an estimator of Z is directly obtained as

$$\hat{Z} = \frac{1}{N}\sum_{i=1}^{N} \psi(S^{(i)}(T)). \tag{3}$$

1.1 Optimal Importance measure as the solution of an optimal control problem

Following section 2.2 of [1], we look for an optimal change of measure by performing a mean-shift on the Brownian motion by a control term $\zeta : [0, T] \times \mathbb{R} \to \mathbb{R}$ that transforms the SDE (1) in

$$\begin{cases} dS_\zeta(t) = [b(t, S_\zeta(t)) + \Sigma(t, S_\zeta(t))\zeta(t, S_\zeta(t))]dt + \Sigma(t, S_\zeta(t))dW(t) \quad t \in (0, T] \\ S_\zeta(0) = S_0. \end{cases} \tag{4}$$

In practice, the sample paths of S_ζ are shifted toward the regions of interest by the control term $\zeta(\cdot,\cdot)$. The quantity of interest (2) can then be rewritten as

$$Z = \mathbb{E}[\psi(S(T))] = \mathbb{E}\Big[\psi(S_\zeta(T)) \exp\Big\{ -\frac{1}{2}\int_0^T |\zeta(t, S_\zeta(t))|^2 dt - \int_0^T \zeta(t, S_\zeta(t)) dW(t) \Big\}\Big]. \quad (5)$$

It turns out that by solving the following Partial Differential Equation (PDE):

$$\begin{cases} \frac{\partial v}{\partial t}(t,s) + b(t,s)\frac{\partial v(t,s)}{\partial s} + \frac{1}{2}\Sigma^2(t,s)\frac{\partial^2 v(t,s)}{\partial s^2} = 0 \quad (t,s) \in [0,T) \times \mathbb{R}^+ \\ v(T,s) = |\psi(s)| \quad s \in \mathbb{R}^+, \end{cases} \quad (6)$$

which corresponds to the Kolmogorov Backward Equation associated with the SDE (1), and computing $\zeta^\star(t,s) = \Sigma(t,s)\frac{\partial \log[v(t,s)]}{\partial s}$ one obtains an estimator $\hat{Z}(\zeta^\star)$ that has zero variance if $\psi \geq 0$ and otherwise has minimal variance i.e. the transformation in the SDE (4) realizes the optimal importance sampling measure. However, computing $\zeta^\star$ requires the solution of the PDE (6), which is not known in closed form, in general. On the other hand, approximate solutions of (6), may be sufficient to obtain substantial variance reduction.

2 European Call Option, general background

We consider a European Call option for which $\psi(S(T)) = e^{-rT}(S(T) - K)_+$, where $(x)_+ = \max(x, 0)$ and we use the Black and Scholes model for the asset price (1):

$$\begin{cases} b(t, S(t)) = rS(t) \\ \Sigma(t, S(t)) = \sigma S(t), \end{cases} \quad (7)$$

where r is the interest rate and σ is the volatility. Under this formulation, (1) admits the solution

$$S(t) = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W(t)}, \quad (8)$$

and a closed form expression for $Z = \mathbb{E}[e^{-rT}(S_T - K)_+]$ is available (Black-Scholes formula):

$$Z = S_0 \Phi[d_1] - e^{-rT} K \Phi[d_2], \quad (9)$$

where $\Phi[\cdot]$ is the cumulative distribution function of a standard Gaussian r.v. and

$$\begin{cases} d_1 = \frac{\log[S_0/K] + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 = d_1 - \sigma\sqrt{T}. \end{cases} \quad (10)$$

Despite the availability of a closed-form solution for the price of a European Call option, in this project we consider an Euler-Maruyama (EM) approximation of the asset price dynamics:

$$S^{m+1} = S^m + b(t_m, S^m)\Delta t + \Sigma(t_m, S^m)\Delta W_m, \quad (11)$$

where S^m approximates $S(t_m)$, with $\{t_m = m\Delta t\}_{m=0}^M$ an equispaced grid in $[0,T]$, and $\Delta W_m = W(t_{m+1}) - W(t_m)$ is the Brownian increment. Our goal is to approximate $Z_{\Delta t} = \mathbb{E}[\psi(S_M)]$. Notice that $Z_{\Delta t}$ will be close to Z for Δt small enough.

Likewise, we consider an Euler-Maruyama (EM) discretization also for the modified dynamics (4) for importance sampling:

$$S_\zeta^{m+1} = S_\zeta^m + [b(t_m, S_\zeta^m) + \Sigma(t_m, S_\zeta^m)\zeta(t_m, S_\zeta^m)]\Delta t + \Sigma(t_m, S_\zeta^m)\Delta W_m, \quad (12)$$

where S_ζ^m is the numerical approximation of $S_\zeta(t_m)$.

2.1 Importance sampling for European Call options

Address the following points:

1. Given an interest rate $r = 5\%$, volatility $\sigma = 0.51$, maturity time $T = 0.2$, initial asset price $S_0 = 100$ and strike price $K = 120$, simulate the trajectories of (1) with the EM discretization based on $M = 100$ subintervals. Compute $Z_{\Delta t}$ with a crude MC estimator and compare the value you obtain with the exact solution (9). Try different samples size $N \in 10, 100, 1000, ...$, what do you observe?

2. Consider the implementation of an Importance Sampling (IS) strategy to approximate $Z_{\Delta t}$, where the importance distribution is constructed by modifying the interest rate r to $\tilde{r}$ in the dynamics (11). Justify your choice of $\tilde{r}$ and quantify the variance reduction obtained.

3. Implement now an adaptive strategy to find the optimal $\tilde{r}$ based on variance minimization. Fix a threshold for the half-width of a confidence interval at level $1 - \alpha$ and iteratively adjust the values of $\tilde{r}$ and increase the sample size N until you achieve the desired tolerance. Compare the previous results with this strategy and quantify the variance reduction obtained.

4. Approximate the solution of equation (6) and compute $\zeta^\star$.

   Plot $\zeta^\star$ on the domain $t \in [0, 0.18]$, $s \in [S_1, S_2]$, where $S_1 = S_0 \exp\{(r - 0.5\sigma^2)T - 3\sigma\sqrt{T}\}$ and $S_2 = S_0 \exp\{(r - 0.5\sigma^2)T + 3\sigma\sqrt{T}\}$. Can you interpret the shape of the control inside this domain?

   Simulate now (12) using the obtained optimal control $\zeta^\star$ and approximate $Z_{\Delta t}$ by importance sampling. Compare the variance reduction obtained with respect to the previous points. Test different levels of space and time discretization of the PDE. How is the variance reduction affected by the discretization size? Is your estimator unbiased?

   Guideline: We recommend solving the PDE (6) with a finite-difference scheme, discretizing implicitly the time variable and using centered finite differences for the first and second order derivatives with respect to the asset price.

   The asset domain has to be truncated; we suggest solving the PDE in the domain $s \in [S_{min}, S_{max}]$ with $S_{min} = S_0 \exp\{(r - 0.5\sigma^2)T - 6\sigma\sqrt{T}\}$ and $S_{max} = S_0 \exp\{(r - 0.5\sigma^2)T + 6\sigma\sqrt{T}\}$. Two meaningful boundary conditions are $v(t, S_{min}) = 0$ and $v(t, S_{max}) = S_{max} - K \exp\{-r(T - t)\}$ for $t \in [0, T]$. As discretization step you can take $\Delta s = (S_{max} - S_{min})/P$ and $\Delta t = T/\tilde{M}$. Start with $P = 50$ and $\tilde{M} = 30$ and monitor the variance reduction as you refine the grid. $\zeta^\star(t, s)$ can be computed by a forward/backward finite difference approximation of $\frac{\partial[\log(v(t,s))]}{\partial s} = \frac{1}{v(t,s)} \frac{\partial v(t,s)}{\partial s}$. Since the solution of the PDE is set to zero on the line $(0, T) \times \{S_{min}\}$, the control may diverge there. Prolongate here the values assumed by the control on $(0, T) \times \{S_{min} + \Delta s\}$. Similarly, the control can be set to zero on the line $\{T\} \times (S_{min}, S_{max})$ since, due to the Forward Euler type discretization (12), it will not affect the asset price. To simulate the controlled trajectories you may need to evaluate the optimal control outside of the grid-points of your finite-difference discretization of the PDE. We suggest considering a piecewise linear interpolation in the domain $[0, T] \times [S_{min}, S_{max}]$ and extending constantly the control outside of this domain.

2.2 Up-and-out European Call option

We consider now the *path-dependent* discounted payoff corresponding to an up-and-out European Call option:

$$\psi(\{S(t)\}_{t\in[0,T]}) = (S(T)-K)_+ e^{-rT} \mathbb{I}_{\{\max\limits_{t\in[0,T]} S(t)\leq U\}}, \tag{13}$$

where $U > 0$ is an externally given barrier and $\mathbb{I}_A$ is the indicator function the set A.

We start considering the following set of parameters: $T = 0.2$, $K = 150$, $\sigma = 0.3$, $U = 200$, $r = 0.1$, $S_0 = 100$. We discretize (1) and (4) through EM with a time-step $\Delta t = T/M$ and $M = 1000$ and we monitor the value of the process only at discrete time instants $t_i = i\Delta t$ for $i = 0, ..., M$. Thus, the quantity to compute is $Z_{\Delta t} = \mathbb{E}[\psi(S_M)\mathbb{I}_{\{\max\limits_{m=0,\ldots,M} S^m \leq U\}}]$.

1. Use a crude MC estimator to compute the price $Z_{\Delta t}$ using different sample sizes and comment your results.

2. Consider the implementation of an importance sampling strategy. Repeat point 3 of section 2.1 in this new framework. Quantify the variance reduction obtained.

3. We now consider the solution of the equation (6) with this new payoff.

   Use a numerical scheme to solve the PDE (6) with final condition $v(T, s) = (s-K)_+e^{-rT}$ and the following boundary conditions:

$$\begin{cases} v(t, S_{min}) = 0 & \text{for } t \in [0, T) \\ v(t, S_{max}) = \epsilon & \text{for } t \in [0, T), \end{cases} \tag{14}$$

   where $S_{max} = U$ and S_{min} follows the same specification as in point 4 of section 2.1 while $\epsilon = 0.1$ is used to avoid numerical instabilities close to the boundary. Notice that, while the boundary condition $v(t, S_{min}) = 0$ arises from the nature of the call option, the boundary condition $v(t, S_{max})$ allows us to take the barrier into account while solving for (6). Compute the optimal control $\zeta^\star$ and, as in section 2.1 point 4, replace the values taken by the control on $(0, T) \times \{S_{min}\}$ with the values it assumes on $(0, T) \times \{S_{min} + \Delta s\}$. Similarly, force to zero the control at the final time instant. Simulate $N = 100$ sample paths of S_ζ according to (12) (as before, interpolate linearly the control inside the domain in which you solved the PDE and extend it constantly outside). Plot the obtained sample paths and comment on what the control action is doing. Finally, compute the option price $Z_{\Delta t}$ by importance sampling using the modified dynamics (12) with $\zeta^\star$. Try different sample sizes and comment on the variance reduction obtained.

4. Repeat the previous points with $U = 170$. In particular, solve the PDE (6) with the following discretization: $P = 3000$, $\tilde{M} = 1500$. What do you notice?

References

[1] Nadhir Ben Rached, Abdul-Lateef Haji-Ali, Shyam Mohan Subbiah Pillai, and Raúl Tempone, *Double-loop importance sampling for McKean–Vlasov stochastic differential equation*, Statistics and Computing **34** (2024), no. 6, 197.