MATH-414 – Stochastic simulation

Lecture 4: Generation of stochastic processes & Monte Carlo Method

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Outline

Continuous time / discrete state Markov chains

- State space $\mathcal{X} = \{y_1, y_2, \dots\}$ (finite or countable isolated points)
- Stochastic process on $\mathcal{X}$: $\{X_t \in \mathcal{X},\ t \geq 0\}$

A process $\{X_t, t \geq 0\}$ is right continuous if, for any realization ω,

$$\lim_{h \to 0^+} X_{t+h}(\omega) = X_t(\omega).$$

A right continuous discrete state Markov process is piecewise constant

- Jump times: $J_0 = 0$,

$$J_n = \inf\{t \geq J_{n-1} : X_t \neq X_{J_{n-1}}\}, \quad n > 0$$

- Holding times:

$$S_n = \begin{cases} J_n - J_{n-1}, & \text{if } J_{n-1} < \infty, \quad n = 1, 2, \dots \\ \infty, & \text{otherwise.} \end{cases}$$

- Jump process: $\{Y_n = X_{J_n},\ n \in \mathbb{N}_0\}$

Poisson process

Definition. *A Poisson process $\{N_t \in \mathbb{N}_0, t \geq 0\}$ with initial state $N_0 = 0$ and parameter $0 < \lambda < \infty$, is a non decreasing, right-continuous, integer valued process which satisfies:*

1. *Independent increments: for all $0 < t_1 < t_2 \leq t_3 < t_4$,*

$$N_{t_2} - N_{t_1} \quad \text{is independent of} \quad N_{t_4} - N_{t_3}$$

2. *Poisson stationary increments: for all $0 < s < t$, $N_t - N_s \sim \mathrm{Pois}(\lambda(t-s))$ i.e.*

$$\mathbb{P}\left(N_t - N_s = j\right) = \frac{(\lambda(t-s))^j}{j!} e^{-\lambda(t-s)}.$$

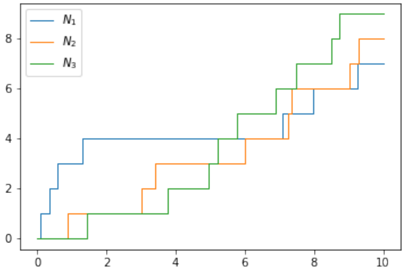
The Poisson process is a (continuous time / discrete state) Markov process. Hence $\{\tilde{N}_t = N_{s+t} - N_s,\ t \geq 0\}$ is also a Poisson process of parameter λ, intependent of $\{N_t,\ t \leq s\}$.

Equivalent characterizations of the Poisson process

a. The holding times $S_1, S_2, \dots$ are independent exponential random variables $\mathrm{Exp}(\lambda)$ and the jump chain is $Y_n = N_{J_n} = n$.

Indeed

- $\mathbb{P}(S_1 > t) = \mathbb{P}(N_t = 0) = e^{-\lambda t} \quad \Rightarrow \quad S_1 \sim \mathrm{Exp}(\lambda)$.
- $\mathbb{P}(S_{n+1} > t) = \mathbb{P}(N_{J_n+t} - N_{J_n} = 0) = e^{-\lambda t}, \quad \Rightarrow \quad S_{n+1} \sim \mathrm{Exp}(\lambda)$
  Moreover, S_{n+1} is independent of $S_1, \dots, S_n$ by independence of increments of N_t.



Equivalent characterizations of the Poisson process

b. For any $t > 0$ and $h \to 0^+$, uniformly in t it holds

$$\begin{aligned}
\mathbb{P}\left(N_{t+h} - N_t = 0\right) &= 1 - \lambda h + o(h), \\
\mathbb{P}\left(N_{t+h} - N_t = 1\right) &= \lambda h + o(h), \\
\implies \mathbb{P}\left(N_{t+h} - N_t > 1\right) &= o(h).
\end{aligned}$$

c. Conditional on $N_t = n$, the n jump times are uniformly distributed in $(0, t)$, i.e. $J_1, \dots, J_n$ have the same distribution of the order statistics $U_{(1)}, \dots U_{(n)}$ with $U_i \overset{\text{iid}}{\sim} \mathcal{U}(0, t)$.

Generation of a Poisson process

Generation based on property a.

Algorithm: Poisson process – version I.

Set $N_0 = 0,\ J_0 = 0,\ Y_0 = 0$
For $n = 1, 2, \dots,$
    Generate $S_n \sim \mathrm{Exp}(\lambda)$ and set $J_n = J_{n-1} + S_n$
    Set $N_t = N_{J_{n-1}},\ t \in [J_{n-1}, J_n)$ and $N_{J_n} = N_{J_{n-1}} + 1$.

Generation on an interval $[0, T]$ based on procerty c.

Algorithm: Poisson process – version II.

Generate $N_T \sim \mathrm{Pois}(\lambda T)$
Generate $U_1, \dots, U_{N_T} \overset{\text{iid}}{\sim} \mathcal{U}(0, T)$
Order the sample $U_{(1)} < \cdots < U_{(N_T)}$
Set $J_0 = 0,\ J_n = U_{(n)}$, and $N_t = n,\ t \in [J_n, J_{n+1}),\ n = 1, \dots, N_T$

Non homogeneous Poisson process

Definition. $\{N_t,\ t \geq 0, N_0 = 0\}$ *is a non-homogeneous Poisson process with rate* $\lambda : [0, \infty) \to \mathbb{R}_+$ *if it is a right-continuous process with independent increments, such that*

$$\mathbb{P}\left(N_{t+h} - N_t = 0\right) = 1 - \lambda(t)h + o(h),$$
$$\mathbb{P}\left(N_{t+h} - N_t = 1\right) = \lambda(t)h + o(h).$$

Lemma

Distribution of the holding times:

$$F_{n+1}(t) = \mathbb{P}\left(S_{n+1} \leq t\right) = 1 - \exp\left\{-\int_{J_n}^{J_n+t} \lambda(s)\, ds\right\}.$$

Proof of the Lemma

$$
\begin{aligned}
F'_{n+1}(t) &= \lim_{h\to 0} \frac{F_{n+1}(t+h) - F_{n+1}(t)}{h} \\
&= \lim_{h\to 0} \frac{\mathbb{P}\left(t < S_{n+1} \leq t+h\right)}{h} = \lim_{h\to 0} \frac{\mathbb{P}\left(S_{n+1} \leq t+h \mid S_{n+1} > t\right)}{h}(1 - F_{n+1}(t)) \\
&= \lim_{h\to 0} \frac{\mathbb{P}\left(N_{J_n+t+h} > n \mid N_{J_n+t} = n\right)}{h}(1 - F_{n+1}(t)) \\
&= \lim_{h\to 0} \frac{1 - \mathbb{P}\left(N_{J_n+t+h} = n \mid N_{J_n+t} = n\right)}{h}(1 - F_{n+1}(t)) \\
&= \lambda(J_n + t)(1 - F_{n+1}(t))
\end{aligned}
$$

Solving the ODE on F_{n+1} gives the desired result.

Generation of non homogeneous Poisson process

Algorithm: Non-homogeneous Poisson process.

Set $N_0 = 0,\ J_0 = 0,\ Y_0 = 0$
For $n = 1, 2, \dots$
    Generate $S_n \sim F_n(t) = 1 - \exp\left\{ - \int_{J_{n-1}}^{J_{n-1}+t} \lambda(s)\, ds \right\}$
    Set $J_n = J_{n-1} + S_n$,
    Set $N_t = N_{J_{n-1}},\ t \in [J_{n-1}, J_n)$,
    Set $N_{J_n} = N_{J_{n-1}} + 1$

Alternative construction:

- Define $\Lambda(t) = \int_0^t \lambda(s)\, ds$
- Let $\tilde{N}_t$ be a homogeneous Poisson process with rate 1
- Then,

$$N_t = \tilde{N}_t \circ \Lambda = \tilde{N}_{\Lambda(t)}$$

is a non-homogeneous Poisson process with rate $\lambda(t)$.

General continuous time / discrete space Markov chain

Let $\{X_t, t \geq 0\}$ be a continuous time Markov chain on the discrete state space $\mathcal{X} = \{y_1, y_2, \ldots\}$.
Then, $\{X_t\}$ is fully characterized by

- distribution of initial state $X_0 \sim \mu$ (with μ a pmf on $\mathcal{X}$)
- the transition probabilities (jump rates)

$$q_{ij}(t) = \lim_{h \to 0^+} \frac{\mathbb{P}\left(X_{t+h} = j \mid X_t = i\right)}{h}$$

$$q_i(t) = \lim_{h \to 0^+} \frac{1 - \mathbb{P}\left(X_{t+h} = i \mid X_t = i\right)}{h}$$

The process is homogeneous if q_{ij} and q_i do not depent on t.

Generator of the Markov process: $\quad Q_{ij} = \begin{cases} q_{ij} & i \neq j \\ -q_i & i = j \end{cases}$

- Q is stable if $q_i < \infty$, $\forall i$
- Q is conservative if $q_i = \sum_{j \neq i} q_{ij}$, $\forall i$

General continuous time / discrete space Markov chain

Definition. *A homogeneous continuous time Markov chain* $\{X_t \in \mathcal{X},\ t \geq 0\}$ *with initial state* $X_0 \sim \mu$ *and stable and conservative generator matrix* Q*, is a right-continuous, piecewise constant process denoted* $\mathrm{Markov}\,(\mu, Q)$ *s.t.*

- *the jump process* $\{Y_n = X_{J_n},\ n \in \mathbb{N}_0\}$ *is a discrete time Markov chain with transition probability*

$$\begin{aligned} \pi_{ij} &= \frac{q_{ij}}{q_i}, \quad i \neq j, \quad \pi_{ii} = 0, && \text{if } q_i \neq 0 \\ \pi_{ij} &= 0, \quad i \neq j, \quad \pi_{ii} = 1, && \text{if } q_i = 0. \end{aligned}$$

- *conditional on* $Y_0, Y_1, \ldots, Y_{n-1}$*, the holding times* $S_1, \ldots, S_n$ *are independent random variables,* $S_i \sim \mathrm{Exp}(q_{Y_{i-1}}), \quad i = 1, \ldots, n.$

Generation of cont. time / discrete space Markov chain

Algorithm: $\mathrm{Markov}\,(\mu, Q)$.

Generate $X_0 \sim \mu$ and set $J_0 = 0,\ Y_0 = X_0$
For $n = 1, 2, \ldots$
    Generate $S_n \sim \mathrm{Exp}(-Q_{Y_n Y_n})$ and set $J_n = J_{n-1} + S_n$,
    Generate $Y_{n+1} \sim \pi_{Y_n,\cdot}$
    Set $X_t = Y_n,\ t \in [J_{n-1}, J_n)$, and $X_{J_{n+1}} = Y_{n+1}$

Example – Poisson process

Generator (Q-matrix) of a homogeneous Poisson process of rate $\lambda > 0$:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & \dots \\ 0 & -\lambda & \lambda & \dots \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

Indeed

$$\begin{aligned} q_i = -Q_{ii} &= \lim_{h\to 0^+} \frac{1 - \mathbb{P}\left(N_{t+h} = i \mid N_t = i\right)}{h} = \lambda \\ q_{i,i+1} = Q_{i,i+1} &= \lim_{h\to 0^+} \frac{\mathbb{P}\left(N_{t+h} = i+1 \mid N_t = i\right)}{h} = \lambda \\ q_{i,j} = Q_{i,j} &= 0, \quad j \neq i, i+1. \end{aligned}$$

Meaning of the generator Q

Let

- $p_i(t) = \mathbb{P}(X_t = y_i)$
- $\mathbf{p}(t) = (p_1(t), p_2(t), \dots)$ (row vector)

$$
\begin{aligned}
\frac{dp_j}{dt}(t) &= \lim_{h\to 0^+} \frac{p_j(t+h) - p_j(t)}{h} = \lim_{h\to 0^+} \frac{1}{h}\left(\mathbb{P}(X_{t+h} = y_j) - p_j(t)\right) \\
&= \lim_{h\to 0^+} \frac{1}{h}\Bigg(\sum_{i\neq j} \underbrace{\mathbb{P}(X_{t+h} = y_j \mid X_t = y_i)}_{=q_{ij}(t)h+o(h)}\, p_i(t) \\
&\qquad\qquad + \underbrace{\mathbb{P}(X_{t+h} = y_j \mid X_t = y_j)}_{=1-q_j(t)h+o(h)}\, p_j(t) - p_j(t)\Bigg) \\
&= \sum_{i\neq j} q_{ij}(t)p_i(t) - q_j(t)p_j(t) = \sum_i p_i(t)Q_{ij}(t)
\end{aligned}
$$

$$
\Longrightarrow \qquad \frac{d}{dt}\mathbf{p}(t) = \mathbf{p}(t)Q(t)
$$

EPFL

Setting

- Z: output of a stochastic model
- **Goal**: estimate $\mu = \mathbb{E}[Z]$
- other properties of the distribution of Z could be of interest as well (higher moments, quantiles, ...)

Assumptions:

- distribution of Z not known / not easily computable
- Z can be simulated (by simulating the stochastic process and evaluating its output)
- Typically $Z = \phi(U_1, U_2, \ldots, U_d)$ where $(U_1, \ldots, U_d)$ are all the uniform random variables used to simulate the stochastic process and ϕ represent the simulation algorithm.

Computing the expectation $\mu = \mathbb{E}[Z]$ can be seen as a high-dimensional integration problem

$$\mu = \mathbb{E}[Z] = \int_{[0,1]^d} \phi(u_1, \ldots, u_d) du_1 \ldots du_d$$

Monte Carlo method

The Monte Carlo method simply consists in

- Generating N i.i.d replicas $Z^{(1)}, \ldots, Z^{(N)}$ of Z (by simulation)
- estimating μ by a sample mean estimator

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} Z^{(i)}$$

We assume hereafter that Z has finite second moments

$$\sigma^2 = \mathbb{V}\mathrm{ar}\left[Z\right] < \infty$$

EPFL

Properties of the Monte Carlo estimator

1. $\hat{\mu}_N$ is unbiased (i.e. $\mathbb{E}[\hat{\mu}_N] = \mu$)

$$\mathbb{E}[\hat{\mu}_N] = \frac{1}{N}\sum_{i=1}^{N} \underbrace{\mathbb{E}\left[Z^{(i)}\right]}_{=\mu,\ \forall i} = \mu$$

(expection is taken w.r.t. the joint distribution of the sample $Z^{(1)}, \dots, Z^{(N)}$)

2. $\mathbb{V}\mathrm{ar}[\hat{\mu}_N] = \frac{\sigma^2}{N}$. Indeed:

$$\mathbb{V}\mathrm{ar}[\hat{\mu}_N] = \mathbb{E}\left[(\hat{\mu}_N - \mathbb{E}[\hat{\mu}_N])^2\right] = \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}(Z^{(i)} - \mu)\right)^2\right]$$

$$= \frac{1}{N^2}\sum_{i,j=1}^{N} \mathbb{E}\left[(Z^{(i)} - \mu)(Z^{(j)} - \mu)\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^{N} \underbrace{\mathbb{E}\left[(Z^{(i)} - \mu)^2\right]}_{=\sigma^2\ \forall i \text{ since } Z^{(i)} \text{ are iid}} + \frac{1}{N^2}\sum_{i\neq j} \underbrace{\mathbb{E}\left[(Z^{(i)} - \mu)(Z^{(j)} - \mu)\right]}_{=0 \text{ since } Z^{(i)}, Z^{(j)} \text{ are indept.}} = \frac{\sigma^2}{N}$$

Properties fo the Monte Carlo estimator

3. Almost sure convergence (from Strong Law of Large Numbers since $\mathbb{E}[Z] < \infty$)

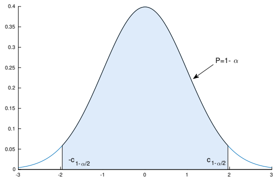
$$\hat{\mu}_N \xrightarrow{N\to\infty} \mu \quad \text{a.s.}$$

4. Asymptotic normality (from Central Limit Theorem since $\mathbb{V}\mathrm{ar}[Z] < \infty$)

$$\frac{\sqrt{N}(\hat{\mu}_N - \mu)}{\sigma} \xrightarrow{\text{d}} \mathcal{N}(0,1) \quad \text{as} \quad N \to \infty$$

Denoting c_α the α-quantile of the standard normal distribution

$$\mathbb{P}\left(\frac{\sqrt{N}|\hat{\mu}_N - \mu|}{\sigma} \leq c_{1-\alpha/2}\right) \xrightarrow{N\to\infty} 1-\alpha$$



$$\implies \qquad |\hat{\mu}_N - \mu| \leq c_{1-\alpha/2}\frac{\sigma}{\sqrt{N}} \quad \text{asympt. with probability } 1-\alpha$$

EPFL

Confidence intervals

Define the asymptotic confidence interval of level $1-\alpha$

$$I_{\alpha,N} = \left[\hat{\mu}_N - c_{1-\alpha/2}\frac{\sigma}{\sqrt{N}},\ \hat{\mu}_N + c_{1-\alpha/2}\frac{\sigma}{\sqrt{N}}\right]$$

Then $\mathbb{P}\left(\mu \in I_{\alpha,N}\right) \xrightarrow{N\to\infty} 1-\alpha$

Problem: $I_{\alpha,N}$ is not directly computable (σ not known in general).
Solution: replace σ^2 with sample variance estimator

$$\hat{\sigma}_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(Z^{(i)} - \hat{\mu}_N\right)^2.$$

Since $\hat{\sigma}_N \to \sigma$ a.s. we have

$$\frac{\sqrt{N}(\hat{\mu}_N - \mu)}{\hat{\sigma}_N} = \underbrace{\frac{\sigma}{\hat{\sigma}_N}}_{\to 1 \text{ a.s.}}\ \underbrace{\frac{\sqrt{N}(\hat{\mu}_N - \mu)}{\sigma}}_{\xrightarrow{\text{d}} \mathcal{N}(0,1)} \xrightarrow{\text{d}} \mathcal{N}(0,1).$$

Computable (approximate) asymptotic confidence interval

$$\hat{I}_{\alpha,N} = \left[\hat{\mu}_N - c_{1-\alpha/2}\frac{\hat{\sigma}_N}{\sqrt{N}},\ \hat{\mu}_N + c_{1-\alpha/2}\frac{\hat{\sigma}_N}{\sqrt{N}}\right]$$

Non asymptotic error bound – Chebyshev

CLT gives only an asymptotic result for $N \to \infty$. For small sample sizes, other more robust bounds can be used.

Bound based on Chebyshev inequality $\mathbb{P}\left(|Y - \mathbb{E}\left[Y\right]| > a\right) \leq \frac{\mathbb{V}\mathrm{ar}[Y]}{a^2}$

Applied to $Y = \hat{\mu}_N$ and with $\mathbb{V}\mathrm{ar}\left[Y\right]/a^2 = \alpha$ gives

$$\mathbb{P}\left(|\hat{\mu}_N - \mu| > \frac{\sigma}{\sqrt{N\alpha}}\right) \leq \alpha$$

Computable (approximate) confidence interval of level $1 - \alpha$

$$\hat{I}^{Cheb}_{\alpha,N} = [\hat{\mu}_N - \frac{1}{\sqrt{\alpha}}\frac{\hat{\sigma}_N}{\sqrt{N}},\ \hat{\mu}_N + \frac{1}{\sqrt{\alpha}}\frac{\hat{\sigma}_N}{\sqrt{N}}].$$

Compare with CLT result $\quad \hat{I}_{\alpha,N} = \left[\hat{\mu}_N - c_{1-\alpha/2}\frac{\hat{\sigma}_N}{\sqrt{N}},\ \hat{\mu}_N + c_{1-\alpha/2}\frac{\hat{\sigma}_N}{\sqrt{N}}\right]$

Notice that $c_{1-\alpha/2} \ll \frac{1}{\sqrt{\alpha}}$ for small α.

Non asymptotic error bound – Berry-Essén

The Berry-Essén bound quantifies the deviation of the cdf of $\frac{\sqrt{N}(\hat{\mu}_N-\mu)}{\sigma}$ from a standard normal cdf Φ – Requires bounded third moments

$$\sup_x \left| \mathbb{P}\left(\frac{\sqrt{N}(\hat{\mu}_N-\mu)}{\sigma} \leq x \right) - \Phi(x) \right| \leq k \frac{\mathbb{E}\left[|Z-\mu|^3\right]}{\sqrt{N}\sigma^3}, \quad (k \approx 0.4748)$$

hence

$$\mathbb{P}\left(\frac{\sqrt{N}|\hat{\mu}_N-\mu|}{\sigma} \leq x \right) \geq \underbrace{2\Phi(x) - 2k \frac{\mathbb{E}\left[|Z-\mu|^3\right]}{\sqrt{N}\sigma^3} - 1}_{\geq 1-\alpha}$$

Given estimates $\hat{\sigma}_N \approx \mathrm{std}[Z]$ and $\hat{\gamma}_{3,N} \approx \mathbb{E}\left[|Z-\mu|^3\right]$, and

$$\hat{x}_\alpha : \qquad \Phi(\hat{x}_\alpha) = 1 - \frac{\alpha}{2} + k \frac{\hat{\gamma}_{3,N}}{\sqrt{N}\hat{\sigma}_N^3} \qquad \text{(corrected quantile)}$$

Computable confidence interval: $\hat{I}_{\alpha,N}^{BE} = [\hat{\mu}_N - \hat{x}_\alpha \frac{\hat{\sigma}_N}{\sqrt{N}},\ \hat{\mu}_N + \hat{x}_\alpha \frac{\hat{\sigma}_N}{\sqrt{N}}]$

Vector valued output

- Output of stochastic model: $\boldsymbol{Z} = (Z_1, \ldots, Z_m)^\top$
- Goal: estimate $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{Z}] = (\mathbb{E}[Z_1], \ldots, \mathbb{E}[Z_m])^\top$

Monte Carlo estimator:

- Generate N iid replicas $\boldsymbol{Z}^{(1)}, \ldots, Z^{(N)}$ of $\boldsymbol{Z}$
- compute $\hat{\boldsymbol{\mu}}_N = \frac{1}{N} \sum_{i=1}^N \boldsymbol{Z}^{(i)}$

Assuming bounded second moments, with covariance matrix $C = \mathbb{E}\left[(\boldsymbol{Z} - \boldsymbol{\mu})(\boldsymbol{Z} - \boldsymbol{\mu})^T\right]$

CLT: $\sqrt{N}(\hat{\boldsymbol{\mu}}_N - \boldsymbol{\mu}) \xrightarrow{\mathrm{d}} \mathcal{N}(0, C)$ and $N(\hat{\boldsymbol{\mu}}_N - \boldsymbol{\mu})^\top C^{-1} (\hat{\boldsymbol{\mu}}_N - \boldsymbol{\mu}) \xrightarrow{\mathrm{d}} \chi^2_m$

C can be replaced by sample covariance matrix $\hat{C}_N = \frac{1}{N-1} \sum_{i=1}^N (\boldsymbol{Z}^{(i)} - \hat{\boldsymbol{\mu}}_N)(\boldsymbol{Z}^{(i)} - \hat{\boldsymbol{\mu}}_N)^\top$

Computable asymptotic confidence region of level $1 - \alpha$

$$\hat{I}_{\alpha,N} = \{\boldsymbol{y} \in \mathbb{R}^m : (\hat{\boldsymbol{\mu}}_N - \boldsymbol{y})^\top \hat{C}_N^{-1} (\hat{\boldsymbol{\mu}}_N - \boldsymbol{y}) \leq \frac{\chi^2_{m;1-\alpha}}{N}\}$$

where $\chi^2_{m;1-\alpha}$ is the $1 - \alpha$ quantile of the χ^2_m distribution.

Delta method

- Output of stochastic model: $\boldsymbol{Z} = (Z_1, \ldots, Z_m)^\top$
- Goal: estimate $\zeta = f(\mathbb{E}[Z_1], \ldots, \mathbb{E}[Z_m])$
  with $f : \mathbb{R}^m \to \mathbb{R}$ a smooth function

Monte Carlo estimator:

- Generate N iid replicas $\boldsymbol{Z}^{(1)}, \ldots, Z^{(N)}$ of $\boldsymbol{Z}$
- compute $\hat{\mu}_{i,N} = \frac{1}{N} \sum_{k=1}^{N} Z_i^{(k)}$
- estimate $\hat{\zeta}_N = f(\hat{\mu}_{1,N}, \ldots, \hat{\mu}_{m,N})$

Notice that in general $\hat{\zeta}$ is biased.

Error estimation can be based on first order Taylor expansion (delta method)

$$\hat{\zeta}_N - \zeta = f(\hat{\boldsymbol{\mu}}_N) - f(\boldsymbol{\mu}) = \nabla f(\boldsymbol{\mu})(\hat{\boldsymbol{\mu}}_N - \boldsymbol{\mu}) + o(\|\hat{\boldsymbol{\mu}}_N - \boldsymbol{\mu}\|).$$

Then

$$\sqrt{N}(\hat{\zeta}_N - \zeta) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \nabla f(\boldsymbol{\mu}) C \nabla f(\boldsymbol{\mu})^\top).$$

Computable asymptotic confidence interval of level $1 - \alpha$

$$\hat{I}_{\alpha,N} = [\hat{\zeta}_N - \Delta_N,\ \hat{\zeta}_N + \Delta_N], \qquad \Delta_N = \frac{c_{1-\alpha/2}}{\sqrt{N}} \sqrt{\nabla f(\hat{\boldsymbol{\mu}}_N) \hat{C}_N \nabla f(\hat{\boldsymbol{\mu}}_N)^\top}$$

Example of Delta method

- Z: output of a stochastic model, with 4 moments bounded
- $Z^{(1)}, \ldots, Z^{(N)} \overset{iid}{\sim} Z$: iid random sample

Let us consider the following standard deviation estimator:

$$\hat{\sigma}_N = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(Z^{(i)})^2 - \left(\frac{1}{N}\sum_{j=1}^{N} Z^{(j)}\right)^2}$$

Defining $Z_1 = Z^2$ and $Z_2 = Z$, we can rewrite as:

$$\sigma = \sqrt{\mathbb{E}[Z^2] - \mathbb{E}[Z]^2} = f(\mathbb{E}[Z_1], \mathbb{E}[Z_2]), \quad f(x,y) = \sqrt{x-y},$$

and $\hat{\sigma}_N = f(\hat{\mu}_{1,N}, \hat{\mu}_{2,N}), \quad \hat{\mu}_{i,N} = \frac{1}{N}\sum_{j=1}^{N} Z_i^{(j)}$

$$\nabla f(\mathbb{E}[Z_1], \mathbb{E}[Z_2]) = (\frac{1}{2\sigma}, -\frac{\mu}{\sigma}), \qquad C = \begin{pmatrix} \mathbb{V}\mathrm{ar}[Z^2] & \mathrm{Cov}(Z^2, Z) \\ \mathrm{Cov}(Z^2, Z) & \mathbb{V}\mathrm{ar}[Z] \end{pmatrix}$$

hence

$$\sqrt{N}(\hat{\sigma}_N - \sigma) \xrightarrow{d} N(0, \gamma), \qquad \gamma = \frac{\mathbb{V}\mathrm{ar}[Z^2]}{4\sigma^2} - \frac{\mu\,\mathrm{Cov}(Z^2, Z)}{\sigma^2} + \mu^2$$