Mock Exam		
EPFL	Stochastic Simulation MATH-414 Prof. F. Nobile **Date:** - **Duration:** 3h	
Sciper:	**Student:**	**Section:**

Exercise 1. Let $\lambda : \mathbb{R}_+ \to \mathbb{R}$ be a strictly positive and bounded function satisfying:

$$0 < \alpha \leq \lambda(t) \leq \lambda_{\max}, \quad \forall t \in [0, \infty).$$

Consider the following function

$$f(t) = \lambda(t) \exp\left(-\int_{t_0}^{t} \lambda(s) ds\right), \quad t \geq t_0.$$

a) Show that f is a probability density function (pdf) and corresponds to the pdf of a shifted exponential distribution $t_0 + \text{Exp}(\bar{\lambda})$ when $\lambda(t) = \bar{\lambda}$, $\forall t \geq t_0$.

b) Consider

$$\lambda(t) = 1 + \eta \cos\left(\frac{2\pi(t - t_0)}{t_1 - t_0}\right), \quad \eta \in (0, 1).$$

Propose and implement a method to generate independent samples from the distribution f for $\eta = 0.5, 0.9$, $t_0 = 0$, $t_1 = 10$ and assess its correctness using qualitative goodness-of-fit tests.

***Hint**: If one needs to solve a nonlinear equation $g(t) = 0$ for $t \in [a, b]$, one can use the bisection method `bisect` or the root finding function `fsolve`, both from the `scipy.optimize` library.*

Exercise 2. A post office is open from time T_1 to time T_2. Customers arrive according to a non-homogeneous Poisson process with rate

$$\lambda_a(t) = 0.1 + \eta\left(1 + \cos\left(\frac{2\pi(t - T_1)}{T_2 - T_1}\right)\right)$$

during opening time $T_1 \leq t \leq T_2$ and $\lambda_a(t) = 0$ beyond the closure time $t \geq T_2$. Their service time is exponentially distributed with constant rate $\lambda_s > 0$. Let N_t denote the length of the queue at time t, with $N_{T_1} = 0$. Under the above assumptions, $\{N_t, t \geq T_1\}$ is a discrete-state, continuous-time Markov process with

$$\begin{aligned}
\lim_{h\to 0^+} \frac{\mathbb{P}(N_{t+h} = j+1 \,|\, N_t = j)}{h} &= \lambda_a(t), \quad \forall j \in \mathbb{N} \\
\lim_{h\to 0^+} \frac{\mathbb{P}(N_{t+h} = j-1 \,|\, N_t = j)}{h} &= \lambda_s, \quad \forall j \in \mathbb{N}, j > 0 \\
\lim_{h\to 0^+} \frac{1 - \mathbb{P}(N_{t+h} = j \,|\, N_t = j)}{h} &= \tilde{\lambda}_a(j,t) = \begin{cases} \lambda_a(t) + \lambda_s & \text{if } j > 0 \\ \lambda_a(t) & \text{if } j = 0 \end{cases}
\end{aligned}$$

Let J_n denote the time of the n-th event (arrival or departure) and $S_n = J_{n+1} - J_n$ the n-th holding time.

a) Suppose $N_{J_n} = j$. Show that the distribution $F_{S_n}(t) = \mathbb{P}(S_n \leq t)$ of the holding time S_n in the two cases $j \neq 0$ and $j = 0$ is

$$F_{S_n}(t) = \int_{J_n}^{t} \tilde{\lambda}_a(j,s) \exp\left(-\int_{J_n}^{s} \tilde{\lambda}_a(j,r)dr\right) ds, \quad t \geq J_n.$$

***Hint**: Notice that*

$$F'_{S_n}(t) = \lim_{h\to 0^+} \frac{\mathbb{P}(t < S_n \leq t+h)}{h} = \lim_{h\to 0^+} \frac{\mathbb{P}(S_n \leq t+h \,|\, S_n > t)\mathbb{P}(S_n > t)}{h} = \ldots.$$

b) Implement an algorithm to simulate the process $\{N_t\}_{t=0}^{\tau}$ where $\tau \geq T_2$ is the first time after closure at which the queue is empty. Consider the values $\eta = \frac{1}{2}$, $\lambda_s = 1$, $T_1 = 0$, $T_2 = 50$. Generate and plot few trajectories of the process.

c) Estimate $\mathbb{E}[\tau]$ by a crude Monte Carlo estimator with a relative error of 1% and a confidence level of 0.95.

d) Assume we now aim at estimating a rare event probability $\mathbb{P}(\tau > 80)$. Propose and discuss in detail (without implementation) a possible variance reduction technique to improve the performance of a crude Monte Carlo estimator for this problem.

Exercise 3. Let $B_R \subset \mathbb{R}^d$ be the d-dimensional ball of radius R in $\mathbb{R}^d$ centered at the origin, and B_R^c its complement. We wish to sample from the following truncated Gaussian distribution

$$f(x) = \frac{1}{Z} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right) \mathbf{1}_{B_R^c}(x),$$

where Z is the normalizing constant and $\mathbf{1}_A$ denotes the indicator function of the set A.

a) Take $d = 2$, $R = 2$, $\mu = (\frac{1}{2}, \frac{1}{2})^T$, and $\Sigma = \begin{bmatrix} 1.9 & 0 \\ 0 & 0.1 \end{bmatrix}$. Implement a Gibbs sampler to sample from $f(x)$. Monitor the convergence using trace plots, autocorrelation functions for the two components X_1 and X_2 of the chain, and a 2D scatter plot. Estimate the effective sample size of this sampler.

Hint: *You can use the following* **`Python`** *commands to produce autocorrelation and scatter plots*

```
import matplotlib.pyplot as plt
import statsmodels.graphics.tsaplots as sm
.
.
.
plt.scatter(X) # To plot scatterplot of X
sm.plot_acf(X_1)  # To plot autocorrelation plot of X_1
```

b) Consider how the covariance matrix Σ affects the performance of the Gibbs sampler. In particular, choose $\Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}$ and comment on the results.

Exercise 4.
Consider the undirected graph in the figure. We focus on the estimation of the expected length ℓ

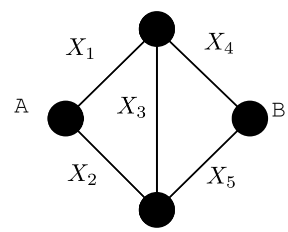

Figure 1: Graph.

of the shortest path between nodes (vertices) A and B, where the lengths of the links (edges) $X_i \sim \mathcal{U}(0, a_i)$, $i = 1, \ldots, 5$, are independent uniform random variables with $(a_1, \ldots, a_5) = (1, 2, 3, 1, 2)$. We have that $\ell = \mathbb{E}[H(\boldsymbol{X})]$ where

$$H(\boldsymbol{X}) = \min\{X_1 + X_4, X_1 + X_3 + X_5, X_2 + X_5, X_2 + X_3 + X_4\}.$$

a) Use a Monte Carlo estimator to compute ℓ with controlled accuracy.

b) Propose and implement a method to compute the sensitivities of ℓ with respect to the parameters $(a_1, \ldots, a_5)$, i.e., estimate $\partial_{a_i}\ell$ at $(a_1, \ldots, a_5) = (1, 2, 3, 1, 2)$.