Stochastic Simulation
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Week of 2025-12-08 to 2025-12-12

Lab 13 of Thursday 11th December 2025

Exercice 1.

Consider the following probability density function in $\mathbb{R}^2$

$$f(x) = \frac{1}{Z}[\exp\{-(1-x_1)^2 - (x_2 - x_1^2)^2\} + \exp\{-(x_1+1)^2 - (x_2 + 3 + x_1^2)^2\}], \quad x = (x_1, x_2)^T \in \mathbb{R}^2,$$

where Z is the (unknown) normalization constant. To generate samples from f, we consider Metropolis-Hastings (MH) type MCMC algorithms. Let us denote with $p(\cdot;\mu,\Sigma)$ [1] the pdf of a $\mathcal{N}(\mu,\Sigma)$ multivariate Gaussian random variable and by

- K_1 the Markov kernel associated to an *Independent Sampler* MH algorithm with proposal density $q_1(x,y) = \frac{1}{2}p(y;(1,1)^T, 0.5\, I_{2\times 2}) + \frac{1}{2}p(y;(-1,-3)^T, 0.5\, I_{2\times 2})$
- K_2 the Markov kernel associated to a *Random Walk* MH algorithm with proposal density $q_2(x,y) = p(y;x,\sigma^2\, I_{2\times 2})$, where $\sigma = 0.15$.

1) Write the explicit expression of the densities corresponding to the Markov kernels K_1 and K_2.
2) Consider now the kernel $K(\omega) = \omega K_1 + (1-\omega)K_2$, with $\omega \in [0,1]$. Show that $K(\omega)$ is a reversible Markov kernel that has f as invariant distribution.
3) Implement a MCMC algorithm that uses the Markov kernel $K(\omega)$ to estimate $\mathbb{E}_f[\|\mathbf{X}\|^2]$, with $\mathbf{X} = (X_1, X_2)^T \sim f(\cdot)$. Include the usual MCMC diagnostic plots in your experiment and describe how you would chose the sample size (length of the chain) to guarantee an error on the computation of $\mathbb{E}_f[\|\mathbf{X}\|^2]$ smaller than $\varepsilon = 1$ with confidence at least .95. Estimate also the expected acceptance rate $\chi(\omega)$ of the implemented algorithm. Try at least two different values for ω.
4) Write a closed-form formula for the expected acceptance rate $\chi(\omega)$ from the expression of the kernel $K(\omega)$. How could one use the results in the previous point to find the value of ω that leads to a target acceptance rate, say $\chi(\omega) = 0.4$?

Hint: You can use the following `Python` commands to plot your results:

```
import statsmodels.graphics.tsaplots as sm
import seaborn as sbn
.
.
.
sbn.kdeplot(x)   # To plot emperical density of samples x
sm.plot_acf(QoI)  # To plot autocorrelation plot of a quantity of interest QoI
```

[1] $p(x;\mu,\Sigma) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$

Exercice 2.

Consider the problem of estimating

$$\frac{d}{d\theta}I(\theta) \tag{2.1}$$

where

$$I(\theta) = \mathbb{E}[\mathbb{1}_{\{\theta X > 1\}}] \tag{2.2}$$

and $X \sim \mathcal{N}(0,1)$. Since the indicator function is discontinuous, we may consider the smoothed version of the integral defined by

$$I_\varepsilon(\theta) = \mathbb{E}\left[\Phi\left(\frac{(\theta X - 1)}{\varepsilon}\right)\right], \tag{2.3}$$

where Φ denotes the CDF of a standard normal random variable. Address the following points.

1) Compute the analytic value of $\frac{d}{d\theta}I(\theta)$ and $\frac{d}{d\theta}I_\varepsilon(\theta)$.
2) Implement the IPA method to compute an approximation of $I_\varepsilon(\theta)$. Using the previously computed values, assess the Monte Carlo error and the error with respect to the smoothing parameter ε.
3) Implement the LR method on the non-regularized function $I(\theta)$.