Stochastic Simulation
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Week of 2025-11-10 to 2025-11-14

Lab 9 of Thursday 13th November 2025

Exercice 1.

Consider the random walk $\{X_n \in \mathbb{Z}, n \in \mathbb{N}_0\}$ with $X_0 \sim \lambda$ on the lattice $\mathcal{X} := \{i \colon i \in \mathbb{Z}, |i| \leqslant 2N^2\}$, whose transition probabilities are given by

$$\mathbb{P}(X_{n+1} = i \pm 1 \big| X_n = i) = \alpha\left(1 \mp \frac{i}{2N^2}\right) , \quad |i| \leqslant 2N^2 ,$$
$$\mathbb{P}(X_{n+1} = i \big| X_n = i) = 1 - 2\alpha ,$$

for some $\alpha \in]0, \frac{1}{2}]$ and $N \in \mathbb{N}$.

1) Implement an algorithm that simulates the Markov Chain $\{X_n \in \mathbb{Z}, n \in \mathbb{N}_0\}$. Use your implementation to address the following points for different values of $N \geqslant 1$:

   a) Assess numerically that the Markov chain converges to an invariant distribution by simulating multiple (independent) chains, each starting in 0 (i.e. $\lambda = \delta_0$). That is, monitor the following quantities (rather, suitable Monte Carlo approximations) as functions of the Markov chain length n.

      i. $\mathbb{E}_\lambda({X_n}^p)^{1/p}$ for $p \in \{1, 2, 4\}$,

      ii. $M_{X_n}(t) := \mathbb{E}_\lambda(e^{tX_n})$ for $t \in [-1, 1]$.

      Speculate on the invariant distribution.

   b) For $N = 10$, compute the eigenvalues and eigenvectors of the transition matrix P. Use the obtained results to deduce the invariant distribution π. **Hint:** Use `np.linalg.eig(P)`.

   c) Assess the validity of the ergodic theorem. That is, verify that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} f(X_n) = \mathbb{E}_\pi(f) , \quad \mathbb{P}_\lambda\text{–a.s.},$$

      for any $f : X \to \mathbb{R}$, with $\sum_n |f(X_n)|\pi_n < \infty$. Specifically, investigate this identity for the moments used in Point 1(a)i and monitor the rate of convergence as a function of n.

2) Consider the rescaled Markov chain $Y_n := \frac{1}{N} X_n$ with state space $\mathcal{Y} := \{x_i \equiv \frac{i}{N} \colon i \in \mathbb{Z}, |i| \leqslant 2N^2\}$. Show by means of numerical simulations that the invariant distribution $\nu \equiv \nu_N$ of $\{Y_n \in \mathbb{Z}, n \in \mathbb{N}_0\}$ is an accurate approximation to the standard normal measure. Moreover, illustrate that the approximation quality improves as N increases.

Exercice 2.

Recall that the standard Metropolis-Hastings algorithm accepts a new candidate state j drawn from the transition matrix Q, given the current state i, with probability $\alpha(i,j) = \min\left(\frac{\pi_j Q_{ji}}{\pi_i Q_{ij}}, 1\right)$, where π is the target probability measure. Consider now a Metropolis-Hastings algorithm that uses the follwing alternative acceptance probabilities

$$\alpha_1(i,j) = \frac{\pi_j Q_{ji}}{\pi_j Q_{ji} + \pi_i Q_{ij}},$$

and

$$\alpha_2(i,j) = \frac{\delta_{ij}}{\pi_i Q_{ij}},$$

with δ such that $\delta_{ij} \leqslant \pi_i Q_{ij} \forall i,j$. Show that, in both cases, the produced Markov chain satisfies the detailed balance condition.