Exam MATH**-410 "Riemann Surfaces"**

18.01.2021

16:15 to 19:15

First name	
Surname	

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6

Total	

Remarks:

- Please fill in your name and sign the exam
- You are not allowed to read the problems until the exam begins
- You have 3 hours for the exam
- There are six problems, written on both sides of the page

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Problem 1. (3+2 points)

Let G be the group of automorphisms of the Riemann sphere generated by the maps

$$z \mapsto \frac{1}{z}, \qquad z \mapsto 1 - z.$$

(a) Show that the quotient is isomorphic to a Riemann surface of genus 0. Find branching points, ramification points and their multiplicities of the natural projection $\mathbb{CP}^1 \to \mathbb{CP}^1/G$.
(b) By the Uniformization Theorem every compact connected Riemann surface of genus 0 is biholomorphic to a Riemann sphere. Show that $\mathbb{CP}^1/G$ is biholomorphic to $\mathbb{CP}^1$ by constructing an explicit biholomorphism.

Problem 2. $(5 + 3 + 3$ points)

Let $\widehat{C}_P$ be the normalized projective curve associated to the homogenous polynomial

$$P(X, Y, Z) = X^3Y + Y^3Z + Z^3X.$$

(a) Compute the genus of $\widehat{C}_P$.
(b) Show that the group $\mathbb{Z}/7\mathbb{Z}$ acts biholomorphically on $\widehat{C}_P$.
(c) Show that the group $\mathbb{Z}/3\mathbb{Z}$ acts biholomorphically on $\widehat{C}_P$.

Remark: In fact, the group $\mathrm{PSL}_2(\mathbb{Z}/7\mathbb{Z})$ acts biholomorphically on $\widehat{C}_P$. The cyclic subroups generated by the elements $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, respectively, have orders 7 and 3, respectively.

Problem 3. (2+2+2+2+2 points)

Check whether the 1-form f is harmonic, holomorphic, exact, closed.
(a) $f = \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}$ on $\mathbb{C} \setminus 0$
(b) $f = y^3\mathrm{d}x + 3xy^2\mathrm{d}y$ on $\mathbb{C} \setminus 0$
(c) $f = \frac{2}{z}\ln(|z|)\mathrm{d}z$ on $\mathbb{C} \setminus 0$
(d) $f = \frac{1}{z}\mathrm{d}z$ on $\mathbb{C} \setminus 0$
(e) $f = e^x \sin(y)\mathrm{d}x + e^x \cos(y)\mathrm{d}y$ on $\mathbb{C}$.

Problem 4. (4 points)

Find all numbers $b \in \mathbb{C}$ such that the Riemann surfaces $X = \mathbb{C} \setminus \{0, 1, 2\}$ and $Y = \mathbb{C} \setminus \{0, 1, b\}$ are biholomorphic.

Hint: Show that a biholomorphism $f : X \to Y$ can be extended to a holomorphic map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$.

Problem 5. (2+2+3+4 points)

Consider the function

$$f(z) = \sum_{m,n\in\mathbb{Z}} \frac{1}{(z + m + in)^4}, \qquad z \in \mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z}).$$

(a) Show that f defines a meromorphic function on the torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

(b) Find poles of f.
(c) Compute the degree of $f : \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \to \mathbb{CP}^1$.
(d) Compute the ramification points of $f : \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \to \mathbb{CP}^1$.

Problem 6. (4 points)

Define an equivalence relation $\sim$ on $\mathbb{C}^*$ by $z_1 \sim z_2$ iff $z_1 = 2^s z_2$ for some $s \in \mathbb{Z}$. Find a lattice $\Lambda \subset \mathbb{C}$ such that the quotient spaces $\mathbb{C}^*/\sim$ and $\mathbb{C}/\Lambda$ are isomorphic as Riemann surfaces.