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Solution 1 We can write $T \stackrel{\text{D}}{=} Z/\sqrt{W/\nu}$, where $Z \sim \mathcal{N}(0,1)$ and $W \sim \chi^2_\nu$ are independent. Hence

$$T^2 \stackrel{\text{D}}{=} \frac{Z^2}{W/\nu} \sim F_{1,n-p},$$

because $Z^2 \sim \chi^2_1$.

Solution 2 The MGF of ε is

$$M_\varepsilon(t) = \mathrm{E}\{\exp(t\varepsilon)\} = \mathrm{E}\{\exp(tX_1 - tX_2)\} = \mathrm{E}\{\exp(tX_1)\}\mathrm{E}\{\exp(-tX_2)\} = M_X(t)M_X(-t),$$

where

$$M_X(t) = \int_0^\infty e^{tx}\lambda e^{-tx}\,\mathrm{d}x = \frac{\lambda}{\lambda - t}, \quad t < \lambda,$$

so

$$M_\varepsilon(t) = \frac{\lambda^2}{\lambda^2 - t^2}, \quad |t| < \lambda.$$

The given density has MGF

$$\frac{\lambda}{2}\int_{-\infty}^\infty e^{tx}e^{-\lambda|x|}\,\mathrm{d}x = \frac{\lambda}{2}\int_0^\infty (e^{-tx-\lambda x} + e^{tx-\lambda x})\,\mathrm{d}x = \frac{\lambda}{2}\left(\frac{1}{\lambda+t} + \frac{1}{\lambda-t}\right) = \frac{\lambda^2}{\lambda^2-t^2}, \quad |t| < \lambda,$$

so it is the MGF of ε.

Clearly $\mathrm{E}(\varepsilon) = \mathrm{E}(X_1) - \mathrm{E}(X_2) = 0$ and $\mathrm{var}(\varepsilon) = 2\mathrm{var}(X_1) = 2/\lambda^2$. Thus we obtain variance σ^2 by setting $\lambda = \sqrt{2}/\sigma$.

This density has heavier tails than the normal, so it might be useful for dealing with data with symmetric errors but large tails than the normal.

Solution 3

(a) If the x_j are all equal then the matrix $X_{n\times 2}$ is not full-rank, so the parameters cannot be identified. If the x_j are not all equal, then

$$X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}, \quad X^{\mathrm{T}}X = \begin{pmatrix} n & n\overline{x} \\ n\overline{x} & \sum x_i^2 \end{pmatrix}, \quad (X^{\mathrm{T}}X)^{-1} = \frac{1}{n\sum(x_i - \overline{x})^2}\begin{pmatrix} \sum x_i^2 & -n\overline{x} \\ -n\overline{x} & n \end{pmatrix},$$

and some algebra using the formulae $\widehat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$ and $H = X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$, so $h_{jj} = (1, x_j)(X^{\mathrm{T}}X)^{-1}(1, x_j)^{\mathrm{T}}$, leads to the given expressions.

(b) Standard formulae for sums of integers and their squares give

$$\sum x_j = c\sum j = cn(n+1)/2, \quad \sum x_j^2 = c^2\sum j^2 = c^2n(n+1)(2n+1)/6,$$

so $\overline{x} = c(n+1)/2$, $\sum(x_j - \overline{x})^2 = c^2n(n+1)(n-1)/12$, clearly h_{jj} is maximised for $j = 1$ and $j = n$, and $x_n - \overline{x} = c(n-1)/2$, giving the stated formula.

(c) This uses the formula for summing a geometric series, i.e., $\sum_{j=1}^n p^j = p(p^n - 1)/(p-1)$ for $p \neq 1$, followed by some algebra.

(d) The sketch is easy. There is limiting normality in (b), but not in (c) (at least in general), because the response at x_n will dominate the limiting distribution. Of course if the errors in (c) were all normal, then there would be exact normality for every n.