Myrto Limnios, Marco Piccininni

Solution 1

a) The assumptions imply that A is injective. If $v \in \mathbb{R}^p \setminus \{0\}$ then

$$v^T Bv = v^T A^T \Omega Av = (Av)^T \Omega(Av) > 0$$

since $Av \neq 0$ and Ω is positive definite. Thus B is positive definite and in particular invertible. The special case $\Omega = I_n$ shows that $A^T A$ is strictly positive definite.

b) Choose $A = (1, 1)^T$ and

$$\Omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then $A^T \Omega A = 0$.

Remark. If Ω has one positive and one negative eigenvalues, we can always find an injective A such that $A^T \Omega A = 0$.

Solution 2 We shall use the following fact. If X_1 and X_2 are indepndent, and Y_1 and X_2 are independent, and X_1 and Y_1 have the same distribution, the for any (measurable) function g, $g(X_1, X_2)$ and $g(Y_1, X_2)$ have the same distribution.

(i) Take $c^T = (1, 0)$

(ii) Take $c^T = (0, 1)$ and use (i). And take $c^T = (-1, 0)$, to get $-X \sim X$, so that $\mathbb{E}[-X] = \mathbb{E}[X]$, and then $\mathbb{E}X = 0$.

(iii) Take $c^T = (1, 1)/\sqrt{2}$. Then

$$X \sim (X + Y)/\sqrt{2} \sim (X_1 + X_2)/\sqrt{2}.$$

(iv) We know that this is true for $n = 1, 2$. Suppose that this is true for n and write

$$(X_1 + \cdots + X_{n+1})/\sqrt{n+1} = \sqrt{n/(n+1)}[(X_1 + \cdots + X_n)/\sqrt{n}] + \sqrt{1/(n+1)}X_{n+1}.$$

This has the same distribution as $\sqrt{n/(n+1)}X + \sqrt{1/(n+1)}Y$ by the induction hypothesis. Now choose $c^T = (\sqrt{n}, 1)/\sqrt{n+1}$

(v) Since X has zero mean and finite variance σ^2, by the central limit theorem

$$(X_1 + \cdots + X_n)/\sqrt{n} \xrightarrow{d} N(0, \sigma^2).$$

By (v) this gives $X \sim N(0, \sigma^2)$, and by (ii) $Y \sim N(0, \sigma^2)$.

(vi) Let $U \sim U[0, 1]$ and

$$(X, Y) = (\cos(2\pi U), \sin(2\pi U)),$$

then by symmetry $c^T(X, Y)$ has the same distribution for all $c \in S^1$ but X and Y are not Gaussian. This is the uniform distribution on the unit circle.

Solution 3

(a) (i), response is y and $X_{n\times 3}$ has rows $(1, 1/x, 1/x^2)$.

(b) (ii), response is y and $X_{n\times 1}$ has rows $1/(1+\beta_1 x)$, with β_1 fixed.

(c) (iii), response is $1/y$ and $X_{n\times 2}$ has rows $(1, x)$.

(d) (ii), response is y and $X_{n\times 2}$ has rows $(1, x^{\beta_2})$, with β_2 fixed.

(e) Can't be done.