Solution 1

(a) We have

$$\begin{aligned}\mathrm{E}\{(\tilde{P}-P)^2\} &= \mathrm{E}\{[\{(\tilde{P}-A\beta)-(P-A\beta)\}^2]\\ &= E\{(\tilde{P}-A\beta)^2\} - 2\mathrm{E}\{(\tilde{P}-A\beta)(A\beta-P)\} + E\{(P-A\beta)^2\}\\ &= \mathrm{var}(\tilde{P}) - 2\mathrm{cov}(\tilde{P},P) + \mathrm{var}(P).\end{aligned}$$

Furthermore,

$$\mathrm{var}(\tilde{P}) = a^{\mathrm{T}}Va, \quad 2\mathrm{cov}(P,\tilde{P}) = 2\mathrm{cov}(P,y)a,$$

which leads to

$$\mathrm{E}\{(\tilde{P}-P)^2\} = a^{\mathrm{T}}Va - 2\mathrm{cov}(P,y)a + \mathrm{var}(P).$$

Hence minimising $\mathrm{E}\{(\tilde{P}-P)^2\}$ with the constraint $\mathrm{E}\{\tilde{P}-P\}=0$ using Lagrange multipliers is equivalent to minimising

$$\begin{aligned}\mathrm{E}\{(\tilde{P}-P)^2\} + \mathrm{E}(\tilde{P}-P)\eta &= a^{\mathrm{T}}Va - 2\mathrm{cov}(P,y)a + 2(a^{\mathrm{T}}X-A)\eta + \mathrm{var}(P)\\ &\equiv a^{\mathrm{T}}Va - 2\mathrm{cov}(P,y)a + 2(a^{\mathrm{T}}X-A)\eta,\end{aligned}$$

where 2η is the vector of Lagrange multipliers (the factor 2 simplifies later algebra). It is straightforward to see that setting the first derivatives with respect to a and η to zero gives the stated optimal solution. Also, you can check that

$$\begin{pmatrix} V & X\\ X^{\mathrm{T}} & 0\end{pmatrix}\begin{pmatrix} V^{-1}-V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}}V^{-1} & V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}\\ (X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}}V^{-1} & -(X^{\mathrm{T}}V^{-1}X)^{-1}\end{pmatrix} = \begin{pmatrix} I & 0\\ 0 & I\end{pmatrix},$$

so the stated inverse is correct, and using it gives

$$a = \left\{V^{-1}-V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}}V^{-1}\right\}\mathrm{cov}(y,P) + V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}A^{\mathrm{T}}. \tag{1}$$

(b) If $P=\beta$, then $A=I$ and $\mathrm{cov}(y,P)=0$, so the optimal a is $V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}$ and $\tilde{P}=a^{\mathrm{T}}y$ is just the usual weighted least squares estimator, which is unbiased and optimal by the Gauss–Markov theorem.

Suppose now that $P=y$, which implies that $A=X$ and $\mathrm{cov}(y,P)=V$. Then

$$a^{\mathrm{T}} = \left\{V^{-1}-V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}}V^{-1}\right\}V + V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}} = I,$$

and thus $\tilde{P}=a^{\mathrm{T}}y=y$, as expected.

(c) In the linear mixed model,

$$\mathrm{var}(\varepsilon') = V = Z\Omega_b Z^{\mathrm{T}} + \Omega.$$

i) If $P=\beta$, then as in b), $\mathrm{cov}(y,P)=\mathrm{cov}(X\beta+\varepsilon',\beta)=0$ and unbiasedness implies that $A=I$. Using equation (**??**), we deduce as before that

$$\tilde{\beta} = a^{\mathrm{T}}y = (X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}}V^{-1}y. \tag{2}$$

ii) If $P = b$, then

$$\operatorname{cov}(y, P) = \operatorname{cov}(X\beta + Zb + \varepsilon, b) = 0 + \operatorname{cov}(Zb, b) + 0 = Z\Omega_b.$$

Furthermore $\operatorname{E}(P) = 0$, which implies that $A = 0$. Equation (**??**) coupled with (**??**) leads to

$$\begin{aligned}\tilde{b} &= a^{\mathrm{T}} y = \Omega_b Z^{\mathrm{T}} \left\{ V^{-1} - V^{-1} X (X^{\mathrm{T}} V^{-1} X)^{-1} X^{\mathrm{T}} V^{-1} \right\} y \\ &= \Omega_b Z^{\mathrm{T}} V^{-1} \left\{ y - X (X^{\mathrm{T}} V^{-1} X)^{-1} X^{\mathrm{T}} V^{-1} y \right\} \\ &= \Omega_b Z^{\mathrm{T}} V^{-1} \left(y - X\tilde{\beta} \right),\end{aligned}$$

as required.

iii) With $P = X\beta + Zb$ we have $A = X$, $\operatorname{cov}(P, y) = Z\Omega_b Z^{\mathrm{T}}$, and inserting these into the formula for a and simplifying yields $\tilde{P} = a^{\mathrm{T}} y = X\tilde{\beta} + Z\tilde{b}$, as expected.