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Problem 1 If $T \sim t_{n-p}$, find the distribution of T^2.

Problem 2 If X_1, X_2 are independent with density $\lambda \exp(-\lambda x)$ for $x > 0$ and $\lambda > 0$, use moment-generating functions to show that $\varepsilon = X_1 - X_2$ has density $(\lambda/2)\exp(-\lambda|\varepsilon|)$ for $\varepsilon \in \mathbb{R}$, and find the moments of ε. Hence say what value of λ ensures that $\text{var}(\varepsilon) = \sigma^2$. Under what circumstances do you think this might be a useful density for the errors in a regression model?

Problem 3 The straight-line regression model has uncorrelated observations $y_j \sim (\beta_0 + \beta_1 x_j, \sigma^2)$, where $j = 1, \ldots, n$ and the x_j are real scalars.

(a) What problem arises if $x_1, \ldots, x_n$ are all equal? If this is not the case, show that

$$\widehat{\beta}_1 = \frac{\sum(x_j - \overline{x})y_j}{\sum(x_j - \overline{x})^2}, \quad \widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1\overline{x}, \quad h_{jj} = \frac{1}{n} + \frac{(x_j - \overline{x})^2}{\sum_i(x_i - \overline{x})^2}, \quad j = 1, \ldots, n,$$

where $\overline{x} = n^{-1}\sum x_j$ and $\overline{y} = n^{-1}\sum y_j$.

(b) If the design points are equally-spaced, i.e., $x_j = cj$ for some non-zero c, show that $\overline{x} = c(n+1)/2$ and $\sum(x_j - \overline{x})^2 = c^2 n(n+1)(n-1)/12$, and deduce that

$$\max_j h_{jj} = h_{nn} = \frac{1}{n} + \frac{3(n-1)}{n(n+1)}.$$

(c) If $x_j = c2^j$ for some non-zero c, show that

$$\overline{x} = c2(2^n - 1)/n, \quad \sum(x_j - \overline{x})^2 = c^2\left\{\frac{4}{3}(4^n - 1) - \frac{4^{n+1} + 4 - 2^{n+3}}{n}\right\},$$

and deduce that $\max_j h_{jj} = h_{nn} \to 3/4$ as $n \to \infty$.

(d) Sketch the designs in (b) and (c), and discuss the limiting distribution of $\widehat{\beta}$ in each case.