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Problem 1 (a) Let $\Omega_{n\times n}$ be a strictly positive definite matrix, and let $A_{n\times p}$ be of rank $p \le n$ (linearly indpendent columns). Show that $B_{p\times p} = A^T \Omega A$ is strictly positive definite, hence invertible. Deduce that $A^T A$ is strictly positive definite and invertible.

(b) Show by a counter example that if Ω is only assumed symmetric and invertible, then B is not necessarily invertible. *Hint: the simplest example for $\Omega_{2\times 2}$ will work here.*

Problem 2 Let $X = (X_1, \dots, X_k)$ be a random vector with finite variance and independent coordinates X_i. We know that

$$X_i \sim N(0, \sigma^2) \text{ for all } i \Longrightarrow c^T X \text{ has the same distribution for any } c \in \mathbb{R}^k \text{ such that } \|c\| = 1$$

We shall see in this assignement that the converse is true for $k = 2$. The proof for any $k \ge 1$ is very similar.

Let $V = (X, Y)^T$ be a random vector in $\mathbb{R}^2$, such that

1. X and Y are independent;
2. $\mathbb{E}X^2 = \sigma^2 < \infty$,
3. $c^T V$ has the same distribution for all $c \in \mathbb{S}^1$.

In the following $X_1, X_2, \dots$ represent independent copies of X.

By a judicious choice of $c \in \mathbb{S}^1$,

(i) show that $c^T V \sim X$, for all $c \in \mathbb{S}^1$.

(ii) show that X and Y have the same distribution and find the expectation of X.

(iii) Show that $X \sim \frac{1}{\sqrt{2}}(X_1 + X_2)$.

(iv) By induction show that the distribution of $n^{-1/2} \sum_{i \le n} X_i$ is the same for all n.

(v) Use the central limit theorem to conclude that $X, Y \sim N(0, \sigma^2)$.

(vi) Show that the above result does not hold if if X and Y are not assumed to be independent. *Hint: For $U \sim$ Unif $(0, 2\pi)$ the distribution of e^{iU} is same as that of $e^{i(U+\phi)}$ for any $\phi \in \mathbb{R}$*

Problem 3 Which of the following can be written as linear regression models, (i) as they are, (ii) when a single parameter is held fixed, (iii) after transformation? For those that can be so written, give the response variable and the form of the design matrix. (a) $y = \beta_0 + \beta_1/x + \beta_2/x^2 + \varepsilon$; (b) $y = \beta_0/(1+\beta_1 x)+\varepsilon$; (c) $y = 1/(\beta_0+\beta_1 x+\varepsilon)$; (d) $y = \beta_0 + \beta_1 x^{\beta_2} + \varepsilon$; (e) $y = \beta_0 + \beta_1 x_1^{\beta_2} + \beta_3 x_2^{\beta_4} + \varepsilon$.