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Problem 1 (Prediction and least squares) Consider prediction of some quantity P, which might be a parameter or a random variable, based on the linear model

$$y \sim (X\beta, V),$$

where $X\beta$ is constant. We aim to choose a predictor $\tilde{P}$ based on y and X that minimises $\mathrm{E}\{(\tilde{P}-P)^2\}$ subject to $\mathrm{E}(\tilde{P}-P)=0$. If we also set $\tilde{P}=a^{\mathrm{T}}y$, then $\tilde{P}$ will be a 'best linear unbiased predictor' (BLUP).

Let $\mathrm{E}(P)=A\beta$ for a clean development; it follows that $a^{\mathrm{T}}X=A$.

(a) Show that

$$\mathrm{E}\{(\tilde{P}-P)^2\} = \mathrm{var}(P) - 2\mathrm{cov}(P,\tilde{P}) + \mathrm{var}(\tilde{P}),$$

and deduce that a may be found by minimising the Lagrangian

$$a^{\mathrm{T}}Va - 2\mathrm{cov}(P,y)a + 2(a^{\mathrm{T}}X - A)\eta.$$

Hence show that the optimal a and η satisfy

$$\begin{pmatrix} V & X \\ X^{\mathrm{T}} & 0 \end{pmatrix}\begin{pmatrix} a \\ \eta \end{pmatrix} = \begin{pmatrix} \mathrm{cov}(y,P) \\ A^{\mathrm{T}} \end{pmatrix},$$

verify that the matrix on the left has inverse

$$\begin{pmatrix} V^{-1} - V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}}V^{-1} & V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1} \\ (X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}}V^{-1} & -(X^{\mathrm{T}}V^{-1}X)^{-1} \end{pmatrix},$$

and that the optimal a is therefore

$$a = \left\{V^{-1} - V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}}V^{-1}\right\}\mathrm{cov}(y,P) + V^{-1}X(X^{\mathrm{T}}V^{-1}X)^{-1}A^{\mathrm{T}}.$$

(b) Check the correctness of this computation when $P=\beta$ and $P=y$.

(c) Now consider the linear mixed model $y = X\beta + Zb + \varepsilon = X\beta + \varepsilon'$, say, and obtain V in this case.

Show that if we take (i) $P=\beta$, then $\mathrm{cov}(y,P)=0$, $A=I$, and (ii) $P=b$, then $\mathrm{cov}(y,P) = Z\Omega_b$, $A=0$. Hence show that

$$\tilde{\beta} = (X^{\mathrm{T}}V^{-1}X)^{-1}X^{\mathrm{T}}V^{-1}y, \quad \tilde{b} = \Omega_b Z^{\mathrm{T}}V^{-1}(y - X\tilde{\beta}).$$

What if (iii) we take $P = X\beta + Zb$?