Solution 1 We write $G(x) = \exp\{-\Lambda(x)\}$, where $\Lambda(x) = \{1 + \xi(x - \eta)/\tau\}_+^{-1/\xi}$.

(a) This seems reasonable because we seek the value x_p that a block maximum will exceed with probability $1/T$.

The exact result solves $-\Lambda(x_p) = \log(1-1/T)$ and (for $\xi \neq 0$) is $x_p = \eta+\tau[\{-\log(1-/T)\}^{-\xi}-1]/\xi$. For the approximation, note that for large T, $-\log(1 - 1/T) \doteq 1/T$. Substituting this into the formula for x_p gives the desired result.

(b) Now we set $F(x)^m \doteq G(x)$ for large x, so we need to solve $1 - 1/(mT) \doteq G^{1/m}(x_p)$, which gives $G(x_p) \doteq \{1 - 1/(mT)\}^m$ or the further approximation $G(x_p) \doteq e^{-1/T}$. Solving these equations gives the stated formulae. Alternatively we note that for large mT,

$$x_p = \eta+\tau\left([-m\log\{1 - 1/(mT)\}]^{-\xi} - 1\right)/\xi \doteq \eta+\tau[\{-m\times -1/(mT)\}^{-\xi}-1]/\xi = \eta+\tau(T^{\xi}-1)/\xi.$$

(c) The blocks of one week have $m = 7 \times 24$ background observations, and $T = 20 \times 52$, the number of one-week blocks in 20 years. The value of p in terms of background observations is $1/(Tm) = 1/(20 \times 52 \times 7 \times 24)$ (ignoring leap years).

The exact and approximate values from (a) are 10.03007 and 10.03103, and from (b) they are 10.03103 and 10.03103, which are all essentially equal, so the formula used is irrelevant.

Solution 2

(a) If $H(x) = 1 - (1 + \xi x/\sigma)_+^{-1/\xi}$ denotes the generalized Pareto distribution function, then

$$\begin{aligned}
\mathrm{P}(M \le x) &= \mathrm{P}\{\max(X_1,\dots,X_N) \le x\} \\
&= \sum_{n=0}^{\infty} \mathrm{P}\{\max(X_1,\dots,X_N) \le x \mid N = n\}\mathrm{P}(N = n) \\
&= \sum_{n=0}^{\infty} H(x)^n \lambda^n e^{-\lambda}/n! \\
&= \exp\{\lambda H(x) - \lambda\} \\
&= \exp\left\{-\lambda(1 + \xi x/\sigma)_+^{-1/\xi}\right\} \\
&= \exp\left[-\{1 + \xi(x - \eta)/\tau\}_+^{-1/\xi}\right],
\end{aligned}$$

where $\eta = \sigma(\lambda^{\xi} - 1)/\xi$ and $\tau = \sigma\lambda^{\xi}$. This is of GEV form, but note that $M \ge 0$, because all the X are non-negative. Hence this formula applies for $x > 0$, and there is a probability mass of $\mathrm{P}(N = 0) = e^{-\lambda}$ at $x = -\infty$, unlike for the GEV.

To check this, note that if $x \le 0$, then $H(x) = 0$, so

$$\mathrm{P}(M \le x) = \exp\{\lambda H(x) - \lambda\} = \exp(-\lambda).$$

(b) Increases in the number of maxima would correspond to an increase in λ, whereas increases in the individual values would stem from changes in σ and/or ξ. So a comprehensive model in which both λ and σ were allowed to depend on time should shed light on the cause of the increase.

Solution 3 The likelihood has two parts: a binomial density for the N observations thatt exceed u_n, and conditional on this, the density for their sizes. For the first, the n observations are independent and the probability that one of them exceeds u_n is p_u,

$$\mathrm{P}(N = n_u) = \binom{n}{n_u} p_u^{n_u}(1 - p_u)^{n-n_u}$$

and for the second we have

$$\prod_{j=1}^{n_u} h(x_j - u_n), \quad x_1, \ldots, x_{n_u} > u_n,$$

where it can be checked that $h(x-u) = -\dot{\Lambda}(x)/\Lambda(u)$ is the generalized Pareto density.

(a) Since the extremal types theorem holds, as $n \to \infty$ with n_u fixed, we have

$$\Lambda_n(u) = n\{1 - \mathrm{P}(X_j > b_n + a_n u)\} \to \Lambda(u),$$

so

$$(n-k)p_u = \frac{n-k}{n}\Lambda_n(u) \to 1 \times \Lambda(u).$$

Hence

$$L_1 = \{n!/(n-n_u)!n_u!\}p_u^{n_u} \to \Lambda(u)^{n_u}/n_u!, \quad L_2 = (1-p_u)^{n-n_u} = \{1-\Lambda_n(u)/n\}^{n-n_u} \to \exp\{-\Lambda(u)\},$$

as hoped.

(b) Since $\Lambda(u)h(x-u) = \{-\dot{\Lambda}(x)\}$, we see that

$$L_2 L_3 \to \Lambda(u)^{n_u} \prod_{j=1}^{n_u} h(x_j - u) = \prod_{j=1}^{n_u} \{-\dot{\Lambda}(x_j)\}$$

so

$$L \to \frac{1}{n_u!} \exp\{-\Lambda(u)\} \prod_{j=1}^{n_u} \{-\dot{\Lambda}(x_j)\}.$$

As the first term does not depend on the parameters, inferences based on L and on the point process likelihood will be similar for large n.