

Solution 1 We follow the hint, first noting that conditional on \mathcal{P} and supposing that there are events at x_1, \dots, x_n ,

$$\left\{ \int f_1 d\mathcal{P}_1 + \int f_2 d\mathcal{P}_2 \mid \mathcal{P} = \sum_{j=1}^n I(x_j) f_1(x_j) + \{1 - I(x_j)\} f_2(x_j) \right.$$

in distribution. Note how the right-hand-side is a sum of independent variables, and for each j we have that $I(x_j) f_1(x_j) + \{1 - I(x_j)\} f_2(x_j)$ takes the value $f_1(x_j)$ with probability $\gamma(x_j)$ and $f_2(x_j)$ with probability $1 - \gamma(x_j)$. Therefore, using the properties of the exponential function and the independence assumption again, we get

$$\mathbb{E} \left\{ \exp \left(- \int f_1 d\mathcal{P}_1 - \int f_2 d\mathcal{P}_2 \right) \mid \mathcal{P} \right\} = \prod_{j=1}^n \left[\gamma(x_j) e^{-f_1(x_j)} + \{1 - \gamma(x_j)\} e^{-f_2(x_j)} \right].$$

Following the hint again, take \mathcal{A} a compact set containing the support of $f_1 + f_2$. Conditional on $N(\mathcal{A}) = n$, the events X_1, \dots, X_n are independent and identically distributed on \mathcal{A} with density function $\dot{\mu}(x)/\mu(\mathcal{A})$, so

$$\begin{aligned} & \mathbb{E} \left\{ \exp \left(- \int f_1 d\mathcal{P}_1 - \int f_2 d\mathcal{P}_2 \right) \mid N(\mathcal{A}) = n \right\} \\ &= \mathbb{E} \left\{ \mathbb{E} \left\{ \exp \left(- \int f_1 d\mathcal{P}_1 - \int f_2 d\mathcal{P}_2 \right) \mid \mathcal{P} \right\} \mid N(\mathcal{A}) = n \right\} \\ &= \mathbb{E} \left\{ \prod_{j=1}^n \left[\gamma(x_j) e^{-f_1(x_j)} + \{1 - \gamma(x_j)\} e^{-f_2(x_j)} \right] \mid N(\mathcal{A}) = n \right\} \\ &= \left(\int_{\mathcal{A}} \left[\gamma(x) e^{-f_1(x)} + \{1 - \gamma(x)\} e^{-f_2(x)} \right] \dot{\mu}(x) dx / \mu(\mathcal{A}) \right)^n. \end{aligned}$$

For brevity we write B for the integral on the last term in the previous equality. As $N(\mathcal{A})$ is Poisson distributed with canonical parameter $\mu(\mathcal{A})$, the unconditional expectation (using the law of iterated expectations again) is

$$\mathbb{E} \left\{ \exp \left(- \int f_1 d\mathcal{P}_1 - \int f_2 d\mathcal{P}_2 \right) \right\} = \sum_{n=0}^{\infty} \frac{\mu(\mathcal{A})^n}{n!} e^{-\mu(\mathcal{A})} \{B/\mu(\mathcal{A})\}^n = \exp \{B - \mu(\mathcal{A})\},$$

and

$$\begin{aligned} B - \mu(\mathcal{A}) &= \int_{\mathcal{A}} \left[\gamma(x) e^{-f_1(x)} + \{1 - \gamma(x)\} e^{-f_2(x)} - 1 \right] \dot{\mu}(x) dx \\ &= \int_{\mathcal{A}} \left[\gamma(x) e^{-f_1(x)} + \{1 - \gamma(x)\} e^{-f_2(x)} - \gamma(x) - \{1 - \gamma(x)\} \right] \dot{\mu}(x) dx \\ &= - \int_{\mathcal{A}} \left\{ 1 - e^{-f_1(x)} \right\} \gamma(x) \dot{\mu}(x) dx - \int_{\mathcal{A}} \left\{ 1 - e^{-f_2(x)} \right\} \{1 - \gamma(x)\} \dot{\mu}(x) dx \\ &= - \int_{\mathcal{E}} \left\{ 1 - e^{-f_1(x)} \right\} \gamma(x) \dot{\mu}(x) dx - \int_{\mathcal{E}} \left\{ 1 - e^{-f_2(x)} \right\} \{1 - \gamma(x)\} \dot{\mu}(x) dx, \end{aligned}$$

where the last step applies because both $1 - e^{-f_1(x)}$ and $1 - e^{-f_2(x)}$ equal zero outside \mathcal{A} . We recognise the exponential of this product as the product of the Laplace functionals $\mathcal{L}_{\mathcal{P}_1}(f_1) \mathcal{L}_{\mathcal{P}_2}(f_2)$ for two independent Poisson processes on \mathcal{E} with respective intensities $\gamma(x) \dot{\mu}(x)$ and $\{1 - \gamma(x)\} \dot{\mu}(x)$, as required.

For the last part, thinning a Poisson process amounts to just keeping the red points, which gives the result.

Solution 2

- (a) We apply the mapping theorem. The first condition holds: for each $x \in \mathbb{R}^s$, $g^{-1}\{x\}$ is a hyperplane of dimension $d - s < d$, which has Lebesgue measure $\mathcal{L}(g^{-1}\{x\}) = 0$. By the existence of the intensity function $\dot{\mu}$ (so the mean measure μ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d), we have

$$\mu(g^{-1}\{x\}) = \int_{g^{-1}\{x\}} \dot{\mu}(u) \mathcal{L}(du) = 0,$$

so $g(\mathcal{P})$ contains no multiple points (with probability one).

The second condition also holds. Since \mathcal{E} is compact and using the definition of a Poisson process, we have, for all compact $\mathcal{A} \subset \mathcal{E}$, that

$$\mu(g^{-1}\{\mathcal{A}\}) \leq \mu(\mathcal{E}) < \infty.$$

Hence $g(\mathcal{P})$ is a Poisson process on $g(\mathcal{E}) \subset \mathbb{R}^s$ with mean measure

$$\mu(\mathcal{A}) = \mu\{g^{-1}(\mathcal{A})\} = \mu(\{(x_1, \dots, x_d) \in \mathcal{X} : (x_1, \dots, x_s) \in \mathcal{A}\}), \quad \mathcal{A} \in \mathbb{R}^s.$$

- (b) This is a Poisson process: any set \mathcal{A} that does not intersect with \mathcal{E}' has mean number of points $\mu(\mathcal{A})$, where $\mu(\mathcal{A}) = 0$, and any that does has mean number of points $\mu(\mathcal{A}) = |\mathcal{A} \cap \mathcal{E}'|$. Clearly the Poisson axioms are satisfied for disjoint sets that intersect with \mathcal{E}' .

The mapping theorem holds in this direction because the mapping is $x \mapsto (x, 0)$ and this neither makes atoms of probability nor gives infinite measure to any compact set.

- (c) Consider the function

$$g_1(t, x) = t, \quad t \in [0, 1], x \in (0, \infty).$$

Let $0 \leq t \leq 1$. We have

$$\mu_1(\{t\}) = \mu\{g_1^{-1}(\{t\})\} = \mu\{\{t\} \times (0, \infty)\} = 0,$$

by the same argument as in (a); the space \mathcal{X} is not compact here but the argument is the same. Consider now a compact $A = [t_1, t_2]$, where $0 \leq t_1 < t_2 \leq 1$. We have

$$\mu_1(A) = \mu\{g_1^{-1}(A)\} = \mu\{[t_1, t_2] \times (0, \infty)\} = (t_2 - t_1) \times 1 = (t_2 - t_1) < \infty.$$

Hence, applying the mapping theorem, we obtain that $\{T_j\}$ is a Poisson process on $[0, 1]$ with mean measure

$$\mu_1([t_1, t_2]) = t_2 - t_1, \quad 0 \leq t_1 < t_2 \leq 1.$$

Now consider the function

$$g_2(t, x) = x, \quad t \in [0, 1], x \in (0, \infty).$$

For $x \in (0, \infty)$, we have $\mu_2(x) = \mu\{g_2^{-1}(x)\} = \mu([0, 1] \times \{x\}) = 0$ by the same argument as before. Moreover, for a compact $B = [x_1, x_2]$, $0 < x_1 < x_2 < \infty$, $\mu_2(B) = \mu\{g_2^{-1}(B)\} = \mu([0, 1] \times B) < \infty$ (the last inequality can for instance be seen by integrating the intensity function). Finally, let $B = (x, \infty)$ for $x > 0$. We have that

$$\mu_2(B) = \mu\{g_2^{-1}(B)\} = \mu\{[0, 1] \times (x, \infty)\} = 1 \times (1 + \xi x)^{-1/\xi}.$$

Thus, applying the mapping theorem, we obtain that $\{X_j\}$ is a Poisson process with mean measure $\mu_2\{[x, \infty)\} = (1 + \xi x)^{-1/\xi}$, $x > 0$.

Solution 3

- (a) A ball of radius r centred at x is empty with probability $P[N\{B_r(x)\} = 0] = \exp\{-\lambda|B_r(x)|\}$. The nearest event is at least r away iff $N\{B_r(x)\} = 0$, so the corresponding density function is

$$-dP[N\{B_r(x)\} = 0]/dr = \lambda d|B_r(x)|/dr \times \exp\{-\lambda|B_r(x)|\}, r > 0.$$

As events are independent, the void probability and the resulting density would be the same if x was an arbitrary point in space.

(b) We see from (a) that $K(r) = \lambda|B_r(x)|/\lambda = |B_r(x)|$, which equals πr^2 when $D = 2$. Hence $L(r) = r$, departures from which will suggest departures from the Poisson process. It is easier to judge departures from a line than from a quadratic.

(c) The data and simulations produce Figure 1.

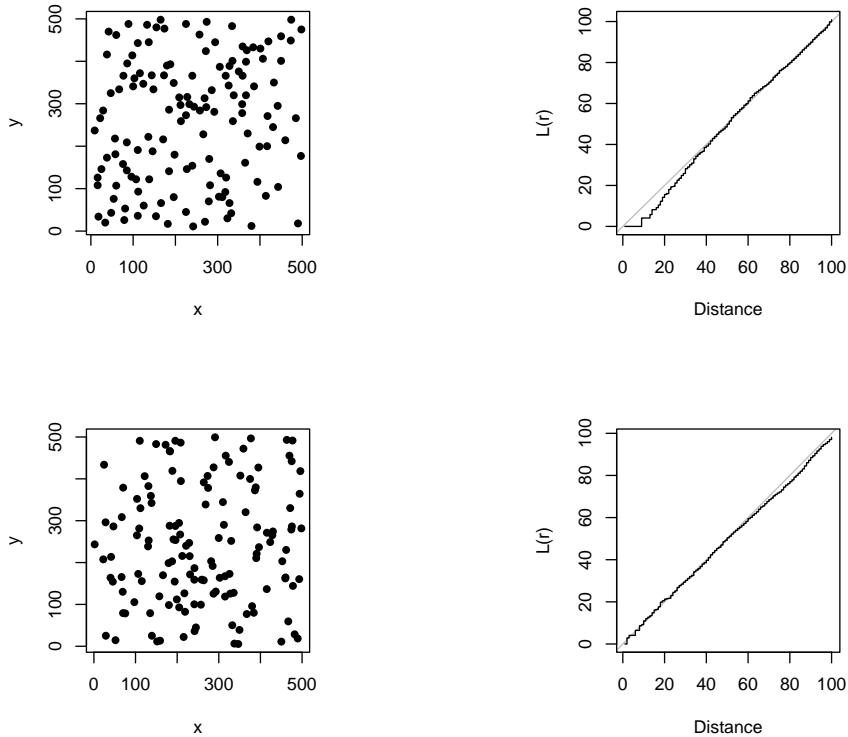


Figure 1: Example output for caveolae analysis and simulated data

The simulation works because the density function for a homogeneous Poisson process on $[0, 500]^2$ is uniform, which is also the case for the density generated by taking $500(U_1, U_2)$ with $U_1, U_2 \stackrel{\text{iid}}{\sim} U(0, 1)$.

The L -function for the data at the upper right panel seems to be too low for small r , which suggests that the caveolae tend not to be as close as the points of a Poisson process. This is confirmed by the lower panels (the simulations show more points close together than do the data) and reinforced by repeated simulations from the model.