Solution 1

(a) Since $G(y) = \exp\{-\Lambda(y)\}$, $G(y)^T = \exp\{-T\Lambda(y)\}$, so we need to consider $T\Lambda(y)$. This equals

$$\begin{aligned} T\left(1+\xi\frac{y-\eta}{\tau}\right)_+^{-1/\xi} &= \left\{1+\left(T^{-\xi}-1\right)+\xi\frac{y-\eta}{\tau T^{\xi}}\right\}_+^{-1/\xi} \\ &= \left\{1+\xi\frac{y-\eta+\tau T^{\xi}(T^{-\xi}-1)/\xi}{\tau T^{\xi}}\right\}_+^{-1/\xi} \\ &= \left\{1+\xi\frac{y-\eta-\tau(T^{\xi}-1)/\xi}{\tau T^{\xi}}\right\}_+^{-1/\xi} \\ &= \left(1+\xi_T\frac{y-\eta_T}{\tau_T}\right)_+^{-1/\xi_T}, \end{aligned}$$

where $\eta_T = \eta + \tau(T^{\xi}-1)/\xi$, $\tau_T = \tau T^{\xi}$ and $\xi_T = \xi$. This proves the result.

(b) We can write

$$G(y;\eta,\tau,\xi) = \exp\left[-\exp\left\{-\frac{1}{\xi}\log\left(1+\xi\frac{y-\eta}{\tau}\right)_+\right\}\right].$$

and since $\log(1+a) = a - a^2/2 + \cdots$ as $a \to 0$, we have

$$\lim_{\xi\to 0} -\frac{1}{\xi}\log\left(1+\xi\frac{y-\eta}{\tau}\right)_+ = -\frac{(y-\eta)}{\tau},$$

as $1+\xi(y-\eta)/\tau > 0$ for small enough ξ and any $(y-\eta)/\tau$. The function $\exp\{-\exp(-x)\}$ is continuous for all x, so

$$\lim_{\xi\to 0} G(y;\eta,\tau,\xi) = \exp\left[-\exp\left\{-(y-\eta)/\tau\right\}\right].$$

Furthermore, $\eta_T = \eta + \tau(T^{\xi}-1)/\xi \to \eta + \tau\log T$, $\tau_T \to \tau$ and $\xi_T \to 0$ as $\xi \to 0$.

(c) As $T \to \infty$, we have $\eta_T \to \begin{cases} +\infty, & \xi > 0 \\ +\infty, & \xi = 0 \\ \eta - \tau/\xi, & \xi < 0 \end{cases}$, $\tau_T \to \begin{cases} +\infty, & \xi > 0 \\ \tau, & \xi = 0 \\ 0, & \xi < 0 \end{cases}$ and $\xi_T \to \xi$.

Increasing T corresponds to taking maxima over a larger block of variables, and the maximum of 100 random variables is always higher than the maximum over only 10 of them. Intuitively, we therefore expect the distribution to shift to the right as we increase T, so the behaviour of η_T makes sense. The behaviour of τ_T is less intuitive, but when $T \to \infty$ and $\xi > 0$ we see that τ_T increases, i.e., the GEV becomes more dispersed, and when $\xi < 0$, $\tau_T \to 0$. i.e., the limiting distribution becomes less dispersed, because the largest values bunch up near the finite upper support point.

(d) The support of $Y \sim G$ corresponds to the set of values $S = \{y : 0 < G(y) < 1\}$.

When $\xi > 0$, $G(y) > 0 \iff \left(1+\xi\frac{y-\eta}{\tau}\right) > 0 \iff y > \eta - \tau/\xi$, so $S = (\eta - \tau/\xi, +\infty)$.

When $\xi < 0$, $G(y) < 1 \iff \left(1+\xi\frac{y-\eta}{\tau}\right) > 0 \iff y < \eta - \tau/\xi$, so $S = (-\infty, \eta - \tau/\xi)$.

When $\xi = 0$, $0 < G(y) < 1$ for all values of $y \in \mathbb{R}$, so $S = \mathbb{R}$.

Solution 2

(a) $X \sim \mathrm{GEV}(0,1,\xi)$ has CDF $G_0(x) = \exp\{-\Lambda_0(x)\}$, say, where $\Lambda_0(x) = (1+\xi x)_+^{-1/\xi}$, and

$$\mathrm{P}(\eta + \tau X \le y) = \mathrm{P}\{X \le (y-\eta)/\tau\} = \exp\left[-\Lambda_0\{(y-\eta/\tau\}\right] = \exp\{-\Lambda(y)\} = \mathrm{P}(Y \le y),$$

and hence $Y \stackrel{\mathrm{D}}{=} \eta + \tau X$. This implies that $\mathrm{E}(Y) = \eta + \tau E(X)$ and $\mathrm{var}(Y) = \tau^2 \mathrm{var}(X)$.

(b) The PDF of X is

$$\frac{\mathrm{d}G_0(x)}{\mathrm{d}x} = \{-\dot{\Lambda}_0(x)\}\exp\{-\Lambda_0(x)\},$$

so

$$\mathrm{E}\left\{X^r G_0(X)^s\right\} = \int x^r \exp\{-s\Lambda_0(x)\}\{-\dot{\Lambda}_0(x)\}\exp\{-\Lambda_0(x)\}\,\mathrm{d}x.$$

If we write $z = (s+1)\Lambda_0(x)$, we need expressions for x and $\{-\dot{\Lambda}_0(x)\}$ in terms of z. The second is easy, because $\mathrm{d}z/\mathrm{d}x = (s+1)\{-\dot{\Lambda}_0(x)\}$, and

$$x = \xi^{-1}\left\{\left(\frac{z}{s+1}\right)^{-\xi} - 1\right\}.$$

Hence

$$\mathrm{E}\left\{X^r G(X)^s\right\} = \frac{1}{s+1}\int_0^\infty \xi^{-r}\left\{\left(\frac{z}{s+1}\right)^{-\xi} - 1\right\}^r e^{-z}\,\mathrm{d}z.$$

Setting $s=0$ and $r=1$, and provided $\xi < 1$ so the gamma function is finite, we have

$$\mathrm{E}(X) = \frac{1}{\xi}\int_0^\infty \left(z^{-\xi} - 1\right)e^{-z}\mathrm{d}z = \frac{1}{\xi}\left(\int_0^\infty z^{-\xi}e^{-z}\mathrm{d}z - 1\right) = \frac{\Gamma(1-\xi) - 1}{\xi}.$$

With $s=0$ and $r=2$, we have, provided $\xi < 1/2$,

$$\begin{aligned}
\mathrm{E}\left(X^2\right) = \frac{1}{\xi^2}\int_0^\infty \left(z^{-\xi} - 1\right)^2 e^{-z}\mathrm{d}z &= \frac{1}{\xi^2}\int_0^\infty \left(z^{-2\xi} - 2z^{-\xi} + 1\right)e^{-z}\mathrm{d}z \\
&= \frac{1}{\xi^2}\left(\int_0^\infty z^{-2\xi}e^{-z}\mathrm{d}z - 2\int_0^\infty z^{-\xi}e^{-z}\mathrm{d}z + 1\right) \\
&= \frac{\Gamma(1-2\xi) - 2\Gamma(1-\xi) + 1}{\xi^2}.
\end{aligned}$$

Note that $\mathrm{E}(X^2)$ exists if and only if $\xi < 1/2$, which is therefore also a condition for $\mathrm{var}(X)$ to be finite. In fact the computations above yield

$$\mathrm{var}(X) = \mathrm{E}\left(X^2\right) - \mathrm{E}(X)^2 = \frac{\Gamma(1-2\xi) - \Gamma(1-\xi)^2}{\xi^2}.$$

Solution 3

(a) Write $(1+\xi x/\sigma)_+^{-1/\xi} = \exp\left\{-\xi^{-1}\log(1+\xi x/\sigma)_+\right\}$ and note that as $\log(1+a) \sim a + o(a)$ when $a \to 0$, $\lim_{\xi\to 0}\xi^{-1}\log(1+\xi x/\sigma)_+ = x/\sigma$ for any $x/\sigma > 0$. As the exponential function is continuous,

$$\lim_{\xi\to 0}\mathrm{P}(X > x) = \lim_{\xi\to 0}\exp\left\{-\xi^{-1}\log(1+\xi x/\sigma)_+\right\} = \exp\left\{-\lim_{\xi\to 0}\xi^{-1}\log(1+\xi x/\sigma)_+\right\} = \exp(-x/\sigma),$$

for $x/\sigma > 0$, as required.

(b) The support is $S_\xi = \{x : f_X(x) > 0\}$, where $f_X(x) = \sigma^{-1}(1+\xi x/\sigma)_+^{-1/\xi - 1}$, with $\sigma > 0$ and $\xi \ne 0$, and $f_X(x) = \sigma^{-1}\exp(-x/\sigma)$ when $\xi = 0$. Hence $S_\xi = \{x : (1+\xi x/\sigma)_+ > 0\}$ when $\xi \ne 0$ and $S_0 = \mathbb{R}_+$. If $\xi > 0$, then $(1+\xi x/\sigma)_+ > 0 \iff (1+\xi x/\sigma) > 0 \iff x > 0$, so $S_\xi = \mathbb{R}_+$. When $\xi < 0$, $(1+\xi x/\sigma)_+ > 0 \iff (1+\xi x/\sigma) > 0 \iff -\sigma/\xi > x > 0$, so in this case $S_\xi = (0, -\sigma/\xi)$. Hence

$$S_\xi = \begin{cases}[0,\infty), & \xi \ge 0,\\ [0, -\sigma/\xi), & \xi < 0.\end{cases}$$

(c) Take $\xi > 0$. Then provided $1 - 1/\xi > 0$, i.e., provided $\xi < 1$, the hint gives

$$\mathrm{E}(X) = \int_0^\infty (1 + \xi x/\sigma)_+^{-1/\xi}\,\mathrm{d}x = \left[\frac{\sigma}{\xi}\frac{1}{1-1/\xi}(1+\xi x/\sigma)_+^{1-1/\xi}\right]_0^\infty = \frac{\sigma}{1-\xi}.$$

It is easy to check that when $\xi \geq 1$ the integral is infinite, so $\mathrm{E}(X)$ is undefined.

When $\xi < 0$ the calculation is the same as above, except that the integral is on the interval $(0, -\sigma/\xi)$, and when $\xi = 0$, $X \sim \exp(1/\sigma)$, so $\mathrm{E}(X) = \sigma$.

If you are curious about the hint, note that

$$\mathrm{E}(X) = \int_0^\infty x f_X(x)\,\mathrm{d}x = \int_0^\infty \int_0^x 1\,\mathrm{d}y f_X(x)\,\mathrm{d}x = \int_0^\infty \int_y^\infty f_X(x)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \mathrm{P}(X > y)\,\mathrm{d}y.$$

(d) We have

$$\mathrm{P}(X > u + x \mid X > u) = \frac{\{1 + \xi(x+u)/\sigma\}_+^{-1/\xi}}{(1+\xi u/\sigma)_+^{-1/\xi}} = \left(\frac{1 + \xi u/\sigma + \xi x/\sigma}{1 + \xi u/\sigma}\right)_+^{-1/\xi} = (1 + \xi x/\sigma_u)_+^{-1/\xi},$$

where $\sigma_u = \sigma + \xi u$. This implies that $X - u \mid X > u \sim \mathrm{GPD}(\xi, \sigma_u)$ and therefore that $\mathrm{E}(X - u \mid X > u) = \sigma_u/(1-\xi) = (\sigma + \xi u)/(1-\xi)$, which is linear in u with intercept $\sigma/(1-\xi)$ and slope $\xi/(1-\xi)$.