Solution 1

(a) We have $\mathrm{P}(a + bU \leq x) = \mathrm{P}\{U \leq (x-a)/b\} = (x-a)/b$, provided that $(x-a)/b \in (0,1)$, i.e., $a < x < a+b$. The corresponding density in this interval is $1/b$, so $a + bU \sim U(a, a+b)$ as required. It follows that $T_j \sim U(0, t_0)$ and $MU_j^* \sim U(0, M)$.

(b) Let R_j denote the event that T_j is retained. This occurs if $MU_j^* < \dot\mu(T_j)$, so $\mathrm{P}(R_j)$ equals

$$\int_0^{t_0} \mathrm{P}(MU_j^* < \dot\mu(t) \mid T_j = t) t_0^{-1}\,\mathrm{d}t = \int_0^{t_0} \mathrm{P}(U_j^* < \dot\mu(t)/M) t_0^{-1}\,\mathrm{d}t = \frac{1}{t_0}\int_0^{t_0} \frac{\dot\mu(t)}{M}\,\mathrm{d}t = \frac{\mu(t_0)}{Mt_0}.$$

Hence the conditional density for T_j, given that it is retained, is

$$f_{T_j}(t \mid R_j) = \frac{\dot\mu(t)/(Mt_0)}{\mu(t_0)/(Mt_0)} = \frac{\dot\mu(t)}{\mu(t_0)}, \quad 0 < t < t_0.$$

We saw in the lectures (Theorem 6, slide 52) that the T_j are therefore a realisation of a Poisson process on $(0, t_0]$ with measure μ. Alternatively we can argue that the T_j are independent, conditional on N, so their joint density is

$$\prod_{j=1}^n \frac{\dot\mu(t_j)}{\mu(t_0)} \times \frac{\mu(t_0)^n}{n!} e^{-\mu(t_0)} = \frac{1}{n!}\prod_{j=1}^n \dot\mu(t_j) e^{-\mu(t_0)}, 0 < t_1, \dots, t_n < t_0,$$

and if we now note that $n!$ permutations of $t_1, \dots, t_n$ would give the same density, we end up with the density of the Poisson process with points at $t_1, \dots, t_n$, as hoped.

(c) The efficiency of the algorithm is clearly $\mu(t_0)/(2Mt_0)$, because the expected number of uniform variables needed to generate the Ts is $2\mathrm{E}(N) = 2Mt_0$ and the expected number of Ts retained is $\mu(t_0)$. Clearly this is minimised if M is as small as possible but always greater than $\dot\mu(t)$.

The algorithm could be improved by finding another function $g(t)$ that satisfies $\dot\mu(t) < g(t) < M$ and from which it is easy to simulate, then retaining T_j in the rejection step if $U_j^* g(T_j) \leq \dot\mu(T_j)$.

Solution 2

(a) The likelihood

$$\exp\{-\mu(t_0)\}\prod_{j=1}^n \dot\mu(t_j) = \exp\left\{-\int_0^{t_0} \exp\left\{\sum_{r=1}^p \beta_r b_r(t)\right\}\mathrm{d}t\right\}\prod_{j=1}^n \exp\left\{\sum_{r=1}^p \beta_r b_r(t_j)\right\}$$

has logarithm

$$\ell(\beta) = \sum_{r=1}^p \beta_r \sum_{j=1}^n b_r(t_j) - \int_0^{t_0} \exp\left\{\sum_{r=1}^p \beta_r b_r(t)\right\}\mathrm{d}t,$$

which is of the given form with $s_r = \sum_{j=1}^n b_r(t_j)$ and $k(\beta) = \int_0^{t_0} \exp\{\sum_{r=1}^p \beta_r b_r(t)\}\,\mathrm{d}t$. This is a linear exponential family with canonical parameters $\beta_1, \dots, \beta_p$ and cumulant generator $k(\beta)$, so the maximum likelihood estimate satisfies the equations $s_r = \partial k(\beta)/\partial\beta_r$, for $r = 1, \dots, p$ and the observed and expected information matrices are equal and have (r, s) element $\partial^2 k(\beta)/\partial\beta_r\partial\beta_s$.

(b) The counts in the successive disjoint intervals $I_k = [(k-1)\Delta, k\Delta)$ $(k = 1, \dots, K)$ are independent Poisson variables, with means $\mu_k = \int_{(k-1)\Delta}^{k\Delta} \dot\mu(t)\,\mathrm{d}t$ that can be approximated by $\Delta\dot\mu\{(k-1/2)\Delta\}$ when Δ is small. The corresponding log likelihood is

$$\sum_{k=1}^K (y_k \log\mu_k - \mu_k - \log y_k!) \equiv \sum_{k=1}^K (y_k \log\mu_k - \mu_k),$$

because additive constants can be dropped from a log likelihood. The approximation arises because $\mu_k \neq \Delta\dot{\mu}\{(k-1/2)\Delta\}$, but provided $\dot{\mu}$ is continuous we can hope that the approximation will be reasonable for K not too large.

(c) If $\dot{\mu}$ is bounded and continuous, the Riemann sum $\sum_{k=1}^{K} \Delta\dot{\mu}\{(k-1/2)\Delta\}$ has limit $k(\beta)$ as $K \to \infty$. Since $y_k = \sum_{j=1}^{n} I(t_j \in I_k)$ counts how many of $t_1, \ldots, t_n$ lie in I_k and no two t_j are equal, for large enough K each of the I_k contains at most one event. If so, then $y_k \log \mu_k$ equals zero if $y_k = 0$ and if $y_k = 1$ then $y_k \log \mu_k = \log \Delta + \log \dot{\mu}\{(k-1/2)\Delta\}$, where $(k-1/2)\Delta$ is the mid-point of the interval that contains t_j. As $K \to \infty$ the series of these midpoints will converge to t_j, and since $\dot{\mu}$ is continuous, this means that $\dot{\mu}\{(k-1/2)\Delta\} \to \dot{\mu}(t_j)$. Putting the pieces together yields the result.

Solution 3

(a) The results here imply that $\widehat{\lambda} = \exp(-0.110184) \doteq 0.896$ and $\widehat{\beta} = 0.00829$, to be compared with 0.881 and 0.00857 on slide 38, so the agreement is not so good.

(b) Setting $K = 101$ gives $\widehat{\lambda} = \exp(-0.118448) \doteq 0.888$ and $\widehat{\beta} = 0.00837$, better but not yet great. Setting $K = 202$ improves matters further, and taking $K = 12 \times 101$ (one-month intervals) gives results that are quite close to the 'exact' results on slide 38 (don't forget that the minimisation giving those might not be perfect).

(c) The code in the question gives

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Call:
glm(formula = y ~ t + offset(log.Delta) + s + c, family = poisson)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.613e-01  1.929e-01  -0.836  0.40289
t            8.028e-03  2.979e-03   2.695  0.00704 **
s           -3.164e-01  1.075e-01  -2.943  0.00325 **
c           -2.965e+12  3.984e+12  -0.744  0.45679
---
Signif. codes:  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 235.20  on 201  degrees of freedom
Residual deviance: 209.01  on 198  degrees of freedom
AIC: 435.45

Number of Fisher Scoring iterations: 5
```

and as the change in residual deviance from the model without the extra two terms is $226.75 - 209.01 = 17.74$, which would be a realisation of a χ_2^2 variable if there was no need for the sine and cosine terms, the significance probability is $\mathrm{P}(\chi_2^2 > 17.74) = 0.00014$, the data strongly suggest that there is an annual cycle in the cyclone arrival times — which of course we would expect.