

Solution 1 Notation is simplified by writing $V = V(z_1, \dots, z_D)$, $V_1 = \partial V(z_1, \dots, z_D) / \partial z_1$, $V_{12} = \partial^2 V(z_1, \dots, z_D) / \partial z_1 \partial z_2$ and so on, and $z'_1 = \partial z_1(y_1) / \partial y_1$ and so forth.

(a) The first derivative is $\partial P(Y_1 \leq y_1, \dots, Y_D \leq y_D) / \partial y_1 = -z'_1 V_1 e^{-V}$, and the second is therefore

$$\frac{\partial P(Y_1 \leq y_1, \dots, Y_D \leq y_D)}{\partial y_1 \partial y_2} = -z'_1 \times z'_2 V_{12} \times e^{-V} - z'_1 V_1 \times -z'_2 V_2 e^{-V} = z'_1 z'_2 (V_1 V_2 - V_{12}) e^{-V},$$

as given in the notes.

(b) The third derivative is

$$z'_1 z'_2 (z'_3 V_{13} \times V_2 + V_1 \times z'_3 V_{23} - z'_3 V_{123}) e^{-V} + z'_1 z'_2 (V_1 V_2 - V_{12}) \times -z'_3 V_3 e^{-V},$$

and reduces to

$$z'_1 z'_2 z'_3 \{-V_1 V_2 V_3 + (V_{12} V_3 + V_{13} V_2 + V_{23} V_1) - V_{123}\} e^{-V}.$$

Recall that the set $\{1, 2, 3\}$ can be partitioned in five different ways, as

$$123, \quad 12 | 3, \quad 13 | 2, \quad 23 | 1, \quad 1 | 2 | 3$$

where the successive partitions have 1, 2, 2, 2, 3 blocks. Thus the term in braces in the third derivative is the sum over all possible partitions of $\{1, 2, 3\}$, with terms having sign $(-1)^b$, where b is the corresponding number of blocks. The same is true in general.

The number of terms in the sum increases very rapidly; there are over 10^5 terms when $D = 10$, so computation of the likelihood (a product of such terms!) quickly becomes infeasible.

Solution 2

(a) We start modelling extremal dependence by fitting the logistic model using the code

```
(ModelD1<-fbvevd(cbind(NO2, PM10), model="log"))
Call: fbvevd(x = cbind(NO2, PM10), model = "log")
Deviance: 9167.191
AIC: 9181.191
Dependence: 0.4369681
```

Estimates

loc1	scale1	shape1	loc2	scale2	shape2	dep
32.4008	10.1389	-0.0286	29.8827	12.4332	0.2925	0.6443

Standard Errors

loc1	scale1	shape1	loc2	scale2	shape2	dep
0.45986	0.31545	0.02202	0.57992	0.48408	0.03487	0.02499

We initially inspect the marginal fit of the model using

```
# marginal fits
par(mfrow=c(2,4)); plot(ModelD1, mar=1)
plot(ModelD1, mar=2)
```

Figure 1 shows reasonable fits for both air pollutants; however the fitted margins tend to respectively underestimate and overestimate the high quantiles of NO2 and PM10.

We now check the bivariate fit using

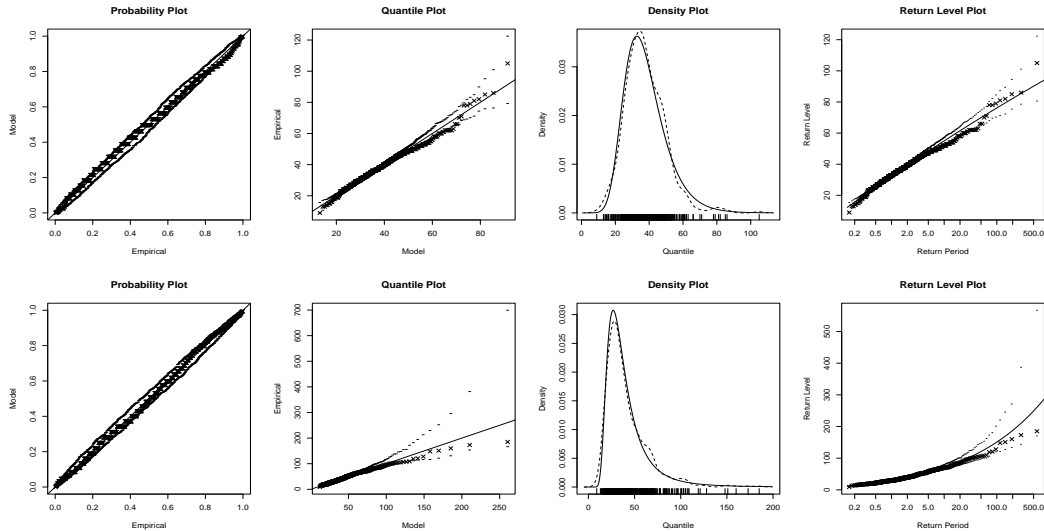


Figure 1: Diagnostic plots of marginal fits for NO2 (top panel) and PM10 (bottom panel).

```
# bivariate fit
par(mfrow=c(2,3)); plot(ModelD1)
```

The fit of the logistic model in Figure 2 seems reasonable. The plot illustrating the Pickands dependence function reveals certain discrepancies between the empirical and the fitted dependence function, which may arise from the presence of asymmetry in extremal dependence.

(b) We now model the bivariate extremal dependence via the asymmetric logistic model using the code

```
(ModelD2<-fbvevd(cbind(NO2, PM10), model="alog"))
Call: fbvevd(x = cbind(NO2, PM10), model = "alog")
Deviance: 9165.467
AIC: 9183.467
Dependence: 0.4418221
```

Estimates

loc1	scale1	shape1	loc2	scale2	shape2	asy1	asy2
32.32013	10.12205	-0.03254	30.02608	12.57328	0.28636	0.99990	0.89178
dep							
0.61477							

Standard Errors

loc1	scale1	shape1	loc2	scale2	shape2	asy1	asy2
4.603e-01	3.152e-01	2.127e-02	5.931e-01	4.948e-01	3.606e-02	2.005e-06	
asy2						dep	
8.331e-02						3.081e-02	

The strength of extremal dependence is similar to that for the logistic model. Figure 3 shows a reasonable fit, although differences between the empirical and fitted Pickands dependence functions subsist.

Since the logistic model is a particular case of the asymmetric logistic, we rely on a likelihood ratio test and compare between the models using the code

```
cp<-ModelD1$dev-ModelD2$dev
```

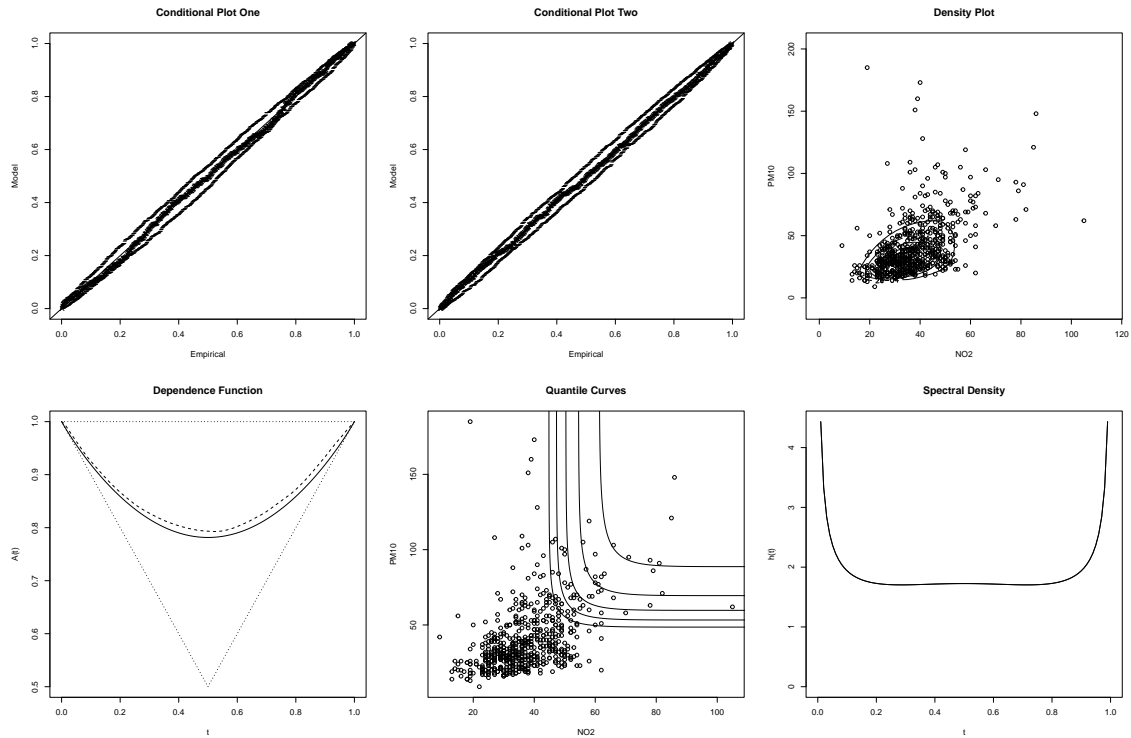


Figure 2: Diagnostic plots of the fit of the bivariate logistic model for daily maxima.

```
1-pchisq(cp, df=2)
0.422309
```

This is larger than 0.05, so it appears that the dependence can be modelled by a logistic dependence structure; there is little evidence for asymmetry,

- (c) We only proceed here with the Hüsler–Reiss model as the fits from other models differ only slightly and do not provide any improvements. We use again the code

```
(ModelD3<-fbvevd(cbind(NO2, PM10), model="hr"))
```

```
Call: fbvevd(x = cbind(NO2, PM10), model = "hr")
Deviance: 9179.674
AIC: 9193.674
Dependence: 0.4168728
```

Estimates

loc1	scale1	shape1	loc2	scale2	shape2	dep
32.35193	10.12590	-0.02004	29.80346	12.43712	0.31536	1.23174

Standard Errors

loc1	scale1	shape1	loc2	scale2	shape2	dep
0.45893	0.31604	0.02024	0.57933	0.48909	0.03347	0.07125

Figure 4 shows a reasonable fit, but both the deviance and AIC for this fit are appreciably larger than those in (a) and (b), indicating that those may be better models.

- (d) In this part we work with weekly maxima of NO2 and PM10, obtained using

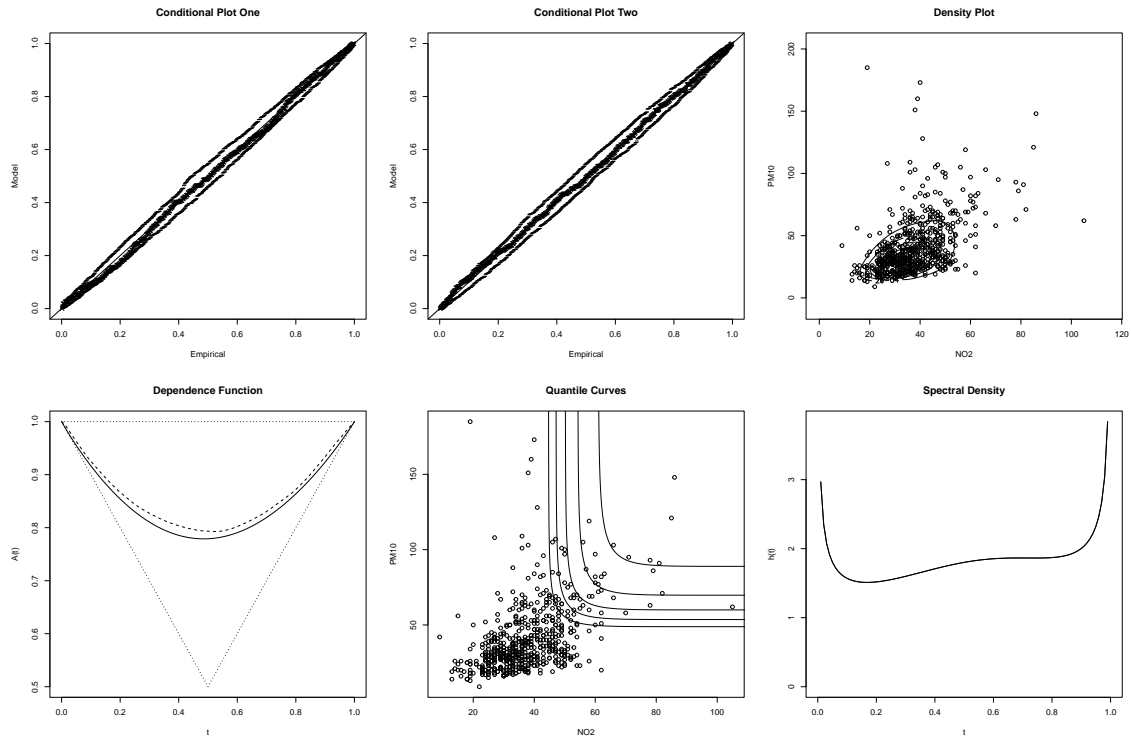


Figure 3: Diagnostic plots of the fit of the bivariate asymmetric logistic model for daily maxima.

```
# We want to split 578 observations into blocks of 7,
# so we initially add zero observations at the end of the vectors
NO2[578:581]<-0; PM10[578:581]<-0;
```

```
NO2_w<-apply(matrix(NO2, ncol=7, byrow=F), 1, function(x)max(x))
PM10_w<-apply(matrix(PM10, ncol=7, byrow=F), 1, function(x)max(x))
```

We model extremal dependence via the logistic dependence structure and obtain the following fit:

```
(ModelW1<-fbvevd(cbind(NO2_w, PM10_w), model="log"))
```

```
Call: fbvevd(x = cbind(NO2_w, PM10_w), model = "log")
```

```
Deviance: 1395.31
```

```
AIC: 1409.31
```

```
Dependence: 0.4604533
```

Estimates

loc1	scale1	shape1	loc2	scale2	shape2	dep
45.5834	9.7346	0.1019	51.4028	20.2904	0.2448	0.6225

Standard Errors

loc1	scale1	shape1	loc2	scale2	shape2	dep
1.20519	0.88295	0.08197	2.57124	2.07558	0.10188	0.06447

The strength of extremal dependence is similar to that for the daily data.

Figure 5 shows a reasonable fit for the dependence structure, although we see again similar problems to the daily data with the Pickands dependence function.

We also check the asymmetric logistic model and obtain

```
(ModelW2<-fbvevd(cbind(NO2_w, PM10_w), model="alog"))
```

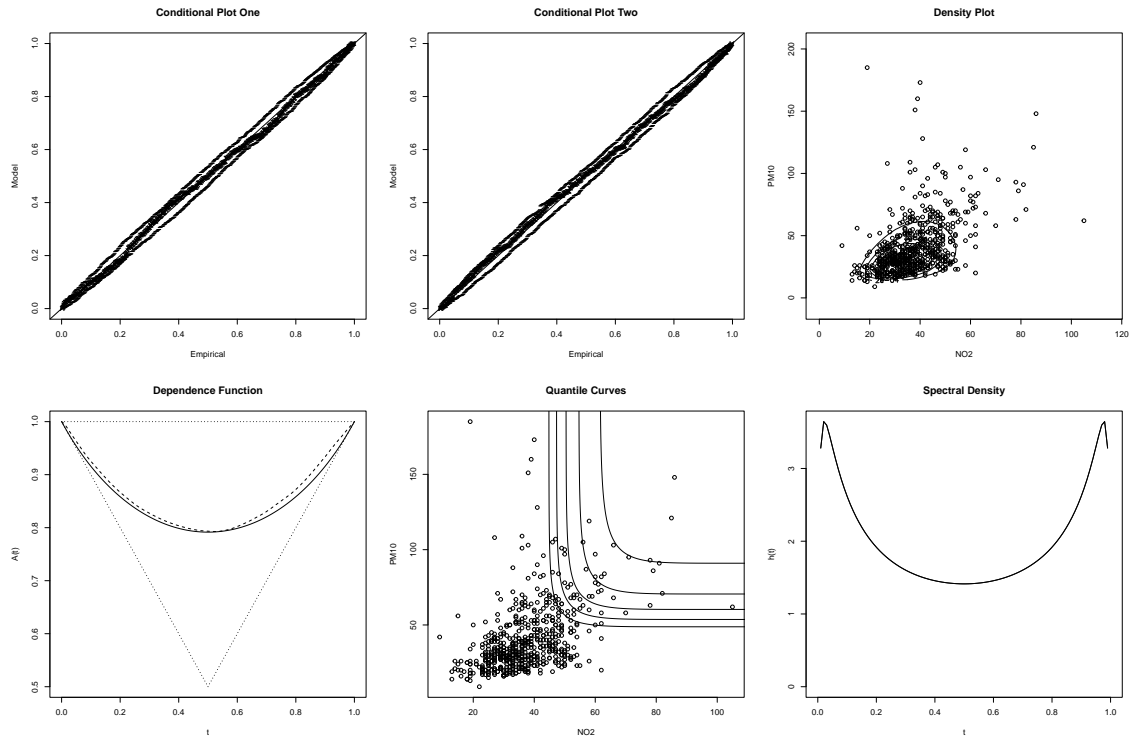


Figure 4: Diagnostic plots of the fit of the bivariate Hüsler–Reiss model for daily maxima.

```
Call: fbvevd(x = cbind(NO2_w, PM10_w), model = "alog")
```

```
Deviance: 1395.579
```

```
AIC: 1413.579
```

```
Dependence: 0.4389836
```

```
Estimates
```

loc1	scale1	shape1	loc2	scale2	shape2	asy1	asy2
45.84077	9.75481	0.09113	51.95000	20.55651	0.22959	0.99944	0.93725
dep							
0.62872							

```
Standard Errors
```

loc1	scale1	shape1	loc2	scale2	shape2	asy1	asy2
1.215e+00	8.841e-01	8.084e-02	2.701e+00	2.116e+00	1.085e-01	2.001e-06	
asy2							dep
3.930e-01							9.031e-02

The fitted Pickands dependence function in Figure 6 fits its empirical counterpart better than for the logistic model; nevertheless the deviances are very similar, so the logistic model still seems reasonable.

Solution 3

- (a) We first fit the logistic dependence structure to the exceedances of NO2 and PM10 using the code

```
airpollutants<-cbind(NO2, PM10)
(POT1 <- evd::fbvpot(airpollutants, apply(airpollutants, 2, quantile, 0.95),
  model="log"))
```

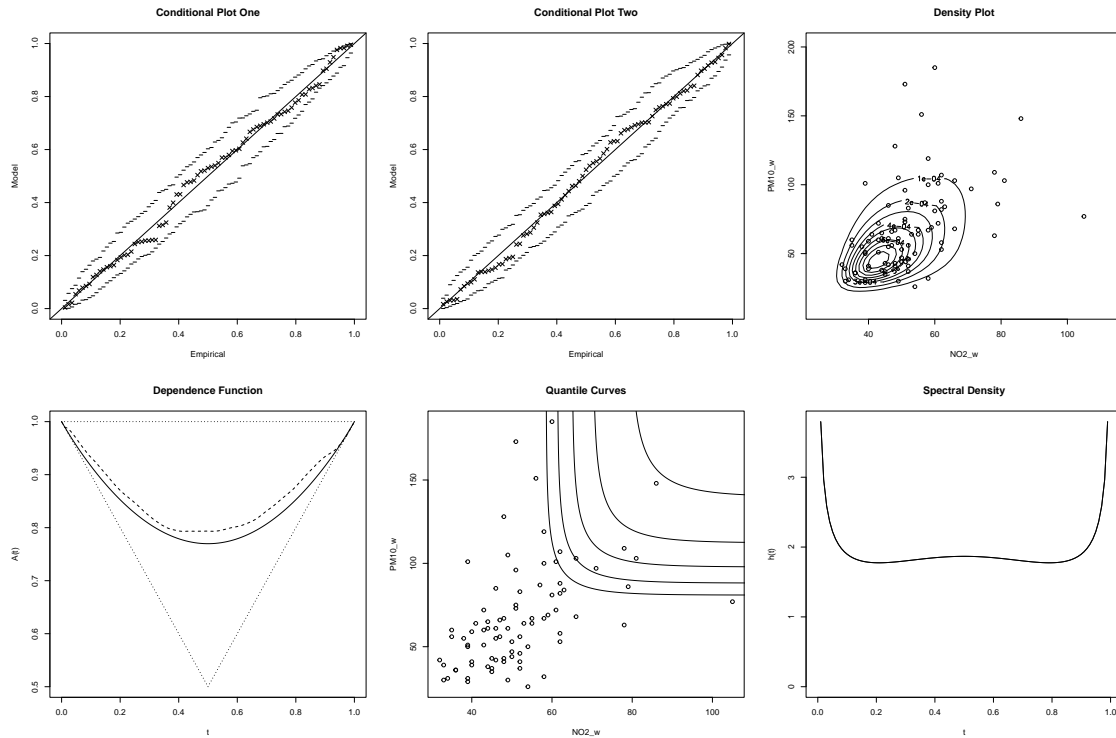


Figure 5: Diagnostic plots of the fit of the bivariate logistic model for weekly maxima.

```

Call: evd::fbvpot(x = airpollutants, threshold = apply(airpollutants, 2, quantile,
  0.95), model = "log")
Likelihood: censored
Deviance: 851.0401
AIC: 861.0401
Dependence: 0.2399396

Threshold: 58 84.15
Marginal Number Above: 26 29
Marginal Proportion Above: 0.045 0.0502
Number Above: 8
Proportion Above: 0.0138

Estimates
  scale1  shape1  scale2  shape2  dep
11.79466  0.01502 26.85926  0.04004  0.81562

Standard Errors
  scale1  shape1  scale2  shape2  dep
3.63755  0.25698  7.86268  0.23691  0.06329

```

The estimated extremal dependence is weaker than for daily maxima, perhaps because selecting a high threshold means that we only consider large exceedances.

We inspect the marginal fits using

```

par(mfrow=c(2,4)); plot(POT1, mar=1)
plot(POT1, mar=2)

```

Figure 7 seems reasonable, in particular for PM10, but problems may arise from the presence of ties and from the small number of exceedances, as shown in the top panel for NO2.

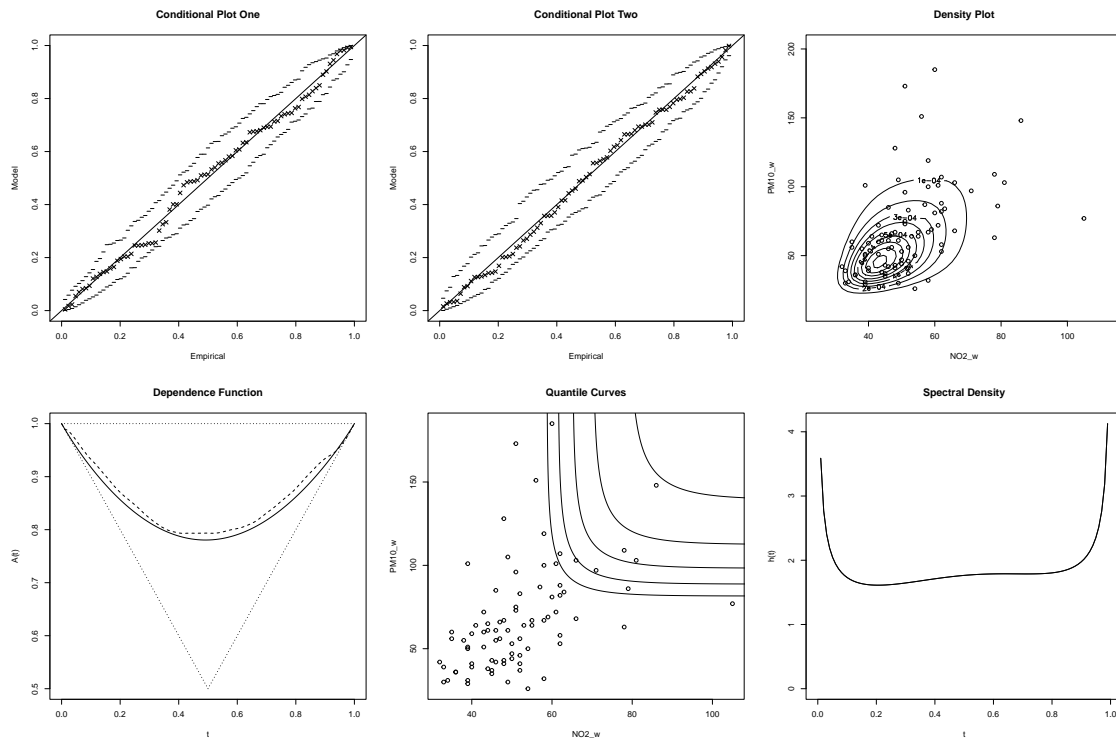


Figure 6: Diagnostic plots of the fit of the bivariate asymmetric logistic model for weekly maxima.

We now check the bivariate fit using

```
par(mfrow=c(1,4)); plot(POT1)
```

Figure 8 shows a reasonable fit for the logistic model, and we notice from the third plot that there are rather few extreme observations jointly in both pollutants.

(b) We repeat the same steps as in part (a) for the asymmetric logistic model using the code

```
(POT2 <- evd::fbvpot(airpollutants, apply(airpollutants, 2, quantile, 0.95),
  model="alog"))
```

```
Call: evd::fbvpot(x = airpollutants, threshold = apply(airpollutants, 2, quantile,
  0.95), model = "alog")
```

```
Likelihood: censored
```

```
Deviance: 842.7556
```

```
AIC: 856.7556
```

```
Dependence: 0.2295116
```

```
Threshold: 58 84.15
```

```
Marginal Number Above: 26 29
```

```
Marginal Proportion Above: 0.045 0.0502
```

```
Number Above: 8
```

```
Proportion Above: 0.0138
```

```
Estimates
```

scale1	shape1	scale2	shape2	asy1	asy2	dep
12.444427	-0.070632	28.987009	0.002408	0.735653	0.242493	0.367385

```
Standard Errors
```

scale1	shape1	scale2	shape2	asy1	asy2	dep
--------	--------	--------	--------	------	------	-----

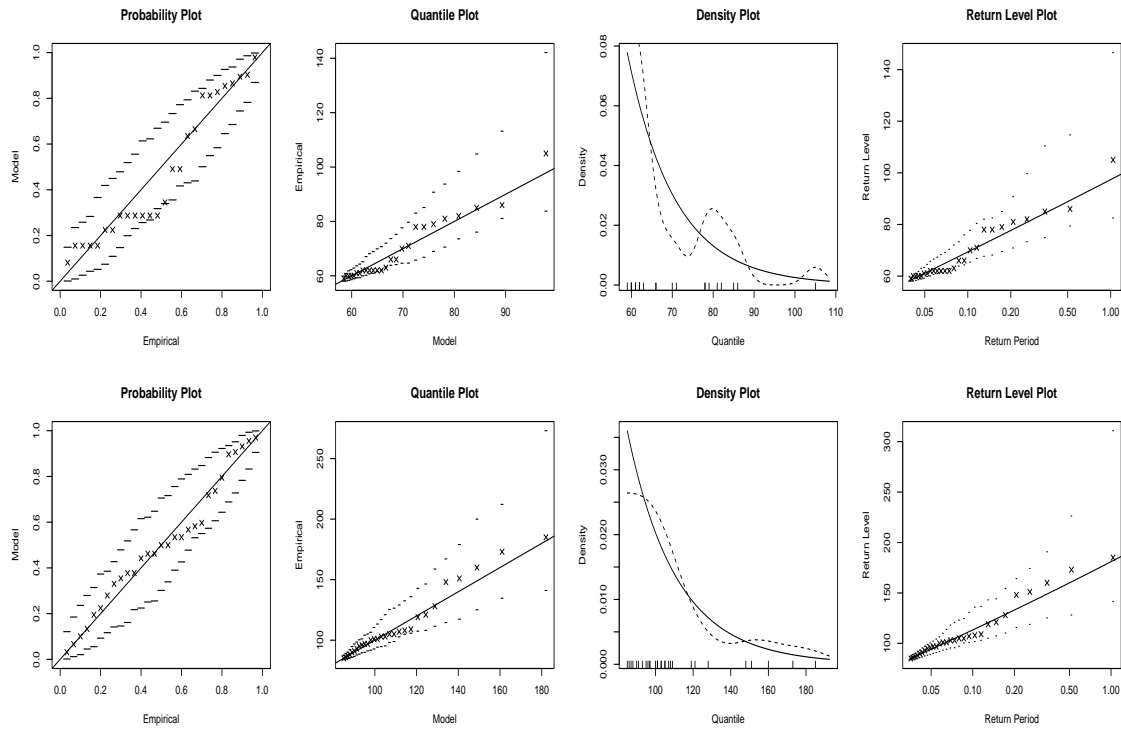


Figure 7: Diagnostic plots of marginal fits for NO2 (top panel) and PM10 (bottom panel).

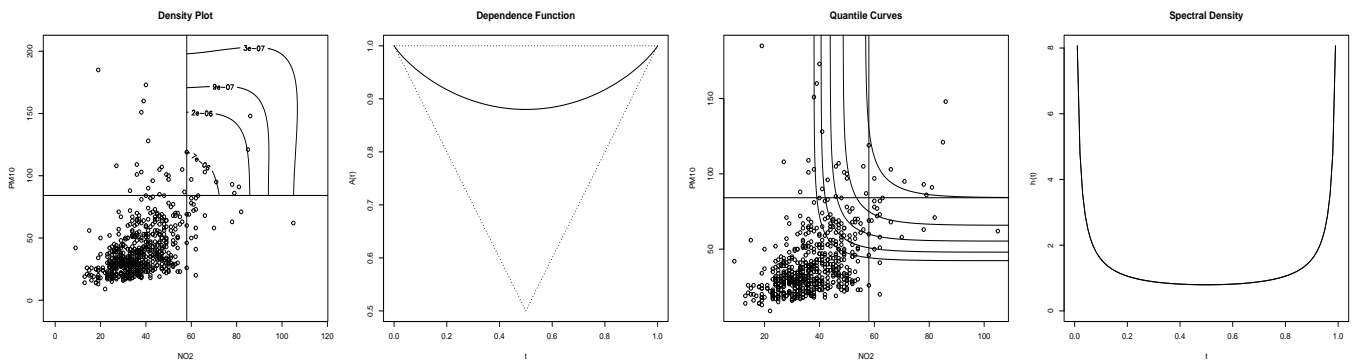


Figure 8: Diagnostic plots of the fit of the bivariate logistic model for exceedances of daily maxima.

3.82401 0.24100 8.66101 0.25969 0.18538 0.08982 0.11088

In contrast to the previous exercise, here we notice that the presence of stronger asymmetry in the fitted model, indicated by the large difference between the `asy1` and `asy2` parameters.

The plots of the bivariate fits in Figure 9 confirm that the fitted dependence structure is highly asymmetric, and the corresponding deviance is much lower than for the logistic model. The likelihood ratio test between the logistic and asymmetric logistic models gives

```
cp2<-POT1$dev-POT2$dev
1-pchisq(cp2, df=2)
0.01588705
```

which suggests a fairly strong departure from symmetry.

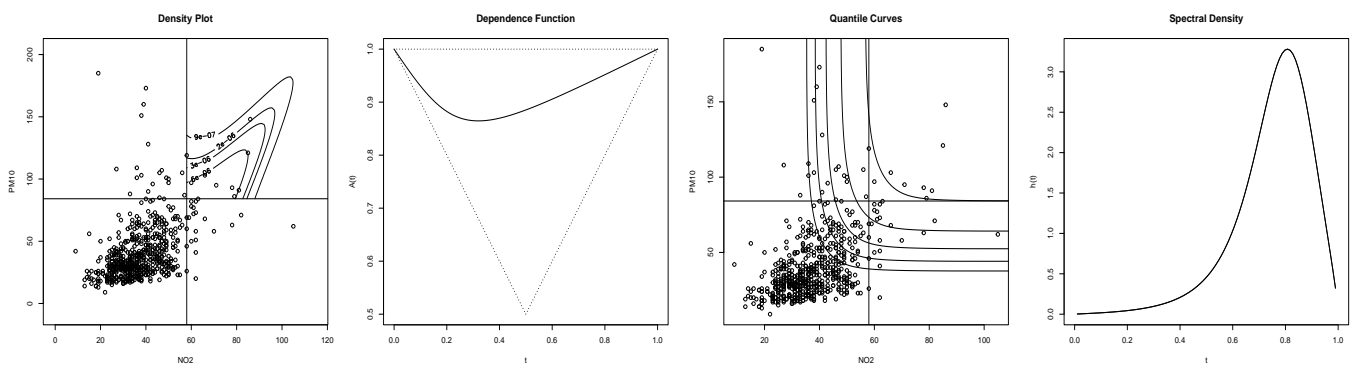


Figure 9: Diagnostic plots of the fit of the bivariate asymmetric logistic model for exceedances of daily maxima.