Solution 1

(a) Since the marginal distributions are unit Fréchet, we must have

$$G(z,\infty,\dots,\infty) = \exp\{-V(z,\infty,\dots,\infty)\} = \exp(-1/z), \quad z>0,$$

i.e., $V(z,\infty,\dots,\infty) = 1/z$, and this holds for any permutation of the arguments. For the max-stability we have

$$\mathrm{P}(Z_d \le b_t + a_t z)^t = [\exp\{-1/(b+t+a_t z)\}]^t = \exp(-1/z)$$

if we set $b_t = 0$ and $a_t = t$, and this obviously holds for every $z, t > 0$.

(b) In the multivariate case we have $a_t = (a_t^1, \dots, a_t^D)$ and $b_t = (b_t^1, \dots, b_t^D)$, say. If

$$G(b_t^1 + a_t^1 z_1, \dots, b_t^D + a_t^D z_D)^t = G(z_1,\dots,z_D), \quad (z_1,\dots,z_D) \in \mathcal{E}^*, t>0,$$

then we see by replacing all but one of the zs by ∞ that we must have $a_t^d = t$ and $b_t^d = 0$ for each d; the marginal distributions must be max-stable with the same choice of as and bs. Therefore

$$\begin{aligned} G(tz_1,\dots,tz_D)^t &= \exp\{-tV(tz_1,\dots,tz_D)\} \\ &= G(z_1,\dots,z_D) = \exp\{-V(z_1,\dots,z_D)\}, \quad (z_1,\dots,z_D)\in\mathcal{E}^*, t>0, \end{aligned}$$

which gives the result.

For the second part, note that

$$\begin{aligned} \mathrm{P}\{\max(Z_1,\dots,Z_D) \le z\} &= \mathrm{P}(Z_1 \le z, \dots, Z_D \le z) \\ &= \exp\{-V(z,\dots,z)\} \\ &= \exp\{-zV(z,\dots,z)/z\} \\ &= \exp\{-V(1,\dots,1)/z\} \\ &= \exp(-\theta_{\mathcal{D}}/z), \quad z>0. \end{aligned}$$

(c) Note that when $z\to\infty$, $\mathrm{P}(Z_1 > z) = 1-\exp(-1/z) = 1/z + O(1/z^2)$ and

$$\begin{aligned} \mathrm{P}(Z_1 > z, Z_2 > z) &= 1 - \mathrm{P}(Z_1 \le z) - \mathrm{P}(Z_2 \le z) + \mathrm{P}(Z_1 \le z, Z_2 \le z) \\ &= 1 - 2\exp(-1/z) + \exp\{-V(z,z)\} \\ &= 1 - 2(1-1/z) + \{1 - V(1,1)/z\} + O(1/z^2) \\ &= (2-\theta)/z + O(1/z^2), \end{aligned}$$

so

$$\chi = \lim_{z\to\infty} \frac{\mathrm{P}(Z_1 > z, Z_2 > z)}{\mathrm{P}(Z_1 > z)} = \lim_{z\to\infty} \frac{(2-\theta)/z + O(1/z^2)}{1/z + O(1/z^2)} = 2-\theta.$$

In the two cases stated we have (independent variables), $\theta = V(1,1) = 1/1 + 1/1 = 2$, so $\chi = 0$, and (totally dependent variables) $\theta = V(1,1) = 1/\min(1,1) = 1$, so $\chi = 1$. Thus θ can be seen as a measure of the asymptotic dependence between the two variables, with smaller values for higher dependence, and $\theta = D$ corresponding to independence.

Solution 2 Setting $z_1 = \infty$ shows that the margin for z_2 is unit Fréchet, and the result for z_1 follows by symmetry. The density function can be written in the form $(z_1 z_2)^{-2} V(z_1,z_2)\exp\{-V(z_1,z_2)\}$, which is positive, so this is a joint density. As $t\to\infty$,

$$F(tz_1, tz_2)^t = \exp\left[-\left\{z_1^{-1} + z_2^{-1} + (tz_1z_2)^{-1}\right\}\right] \to \exp\left\{-\left(z_1^{-1} + z_2^{-1}\right)\right\},$$

so it is not max-stable, and the limiting distribution for rescaled maxima is independence. For it to correspond to a Poisson process intensity, V should correspond to a measure on $\mathcal{E}^*$, so the measure of any rectangle $(z_1', z_1) \times (z_2', z_2)$ must be positive, i.e.,

$$V(z_1, z_2') + V(z_1', z_2) - V(z_1, z_2) - V(z_1', z_2') > 0, \quad 0 < z_1' < z_1, 0 < z_2' < z_2.$$

However this equals $-(1/z_1' - 1/z_1)(1/z_2' - 1/z_2)$, which is negative, so there is no such Poisson process.

Solution 3

(a) The case $\lambda \to \infty$ is straightforward. If $\lambda \to 0$ and $z_2 > z_1$, then $\lambda^{-1}\log(z_2/z_1) \to \infty$ and $\lambda^{-1}\log(z_1/z_2) \to -\infty$, so $V(z_1, z_2) \to 1/z_1 = 1/\min(z_1, z_2)$. Likewise, the limit is $1/z_2 = 1/\min(z_1, z_2)$ if $z_1 > z_2$.

Here $\theta = V(1,1) = 2\Phi(\lambda/2) \in [1, 2)$ tends to 2 when $\lambda \to \infty$ and equals 1 when $\lambda = 0$.

(b) Suppose without loss of generality that $z_1 > z_2$. Then

$$(z_1^\alpha + z_2^\alpha)^{-1/\alpha} = z_1^{-1}\{1 + (z_2/z_1)^\alpha\}^{-1/\alpha},$$

where $1 + (z_2/z_1)^\alpha > 1$, so $\{1 + (z_2/z_1)^\alpha\}^{-1/\alpha} \to 0$ as $\alpha \to 0$. Hence $V(z_1, z_2) \to 1/z_1 + 1/z_2$, corresponding to independence.

Now suppose that $\alpha \to \infty$, and write

$$(z_1^\alpha + z_2^\alpha)^{-1/\alpha} = z_2^{-1}\{1 + (z_1/z_2)^\alpha\}^{-1/\alpha} \to z_2^{-1}\{(z_1/z_2)^\alpha\}^{-1/\alpha} = z_1^{-1}.$$

Hence $V(z_1, z_2) \to 1/z_2 = 1/\min(z_1, z_2)$, as required.

Here $\theta = V(1,1) = 2 - 2^{-1/\alpha} \in (1, 2)$ tends to 2 when $\alpha \to 0$ and tends to 1 when $\alpha \to \infty$.