Solution 1

(a) We first simulate the moving maximum process with Fréchet margins from the code below and inspect the plots in Figure 1.

```
n <- 10000; a <- 1; i <- c(1:n)   # we saw this before
z <- 1/rexp(n+1)   # independent Frechet variables
x <- pmax(a*z[i],z[i+1])/(a+1)  # moving maximum series
par(mfrow=c(1,2))  # two adjacent panels for figures

chi.lag <- function( x, lag=0)  chiplot(cbind(x[1:(n-lag)],x[(1+lag):n]), which=1)
chi.lag( x, 1)
chi.lag( x, 2)
```

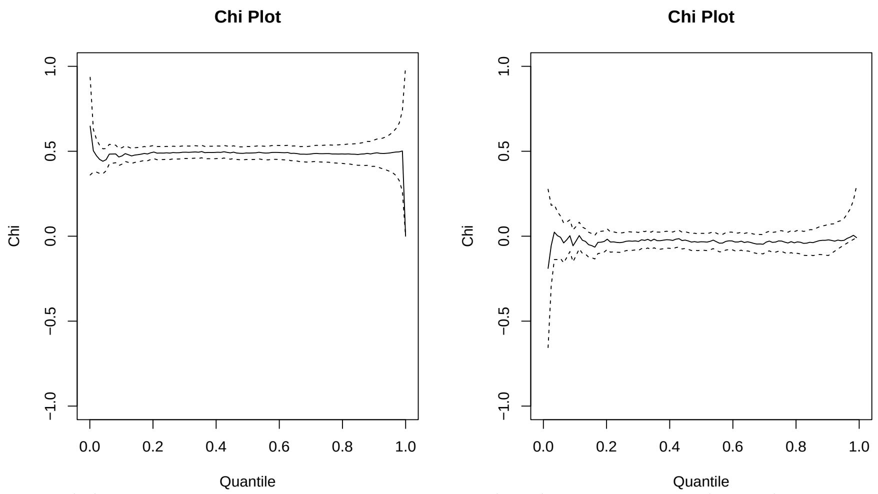

Figure 1: Plots of $\chi_h(u)$ for $a = 1$ and the lags $h = 1$ (left) and $h = 2$ (right) for the moving maximum process.

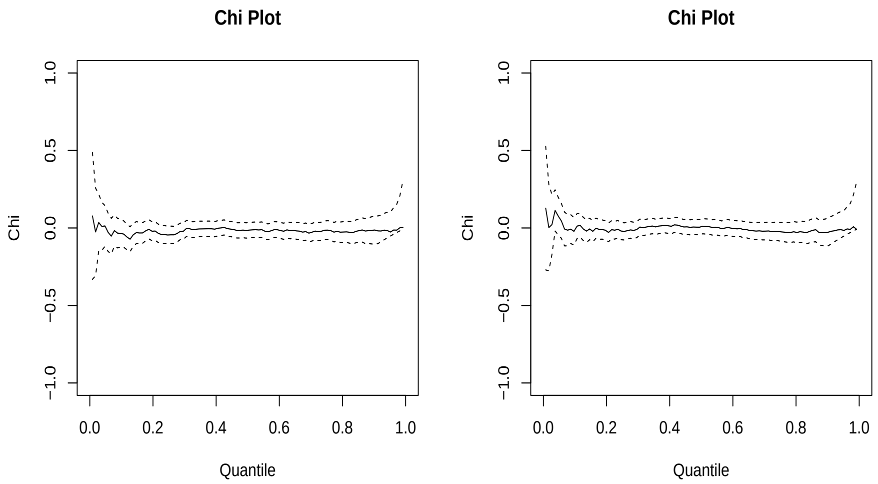

Figure 2: Plots of $\chi_h(u)$ for $a = 1$ and the lags $h = 3$ (left) and $h = 4$ (right) for the moving maximum process.

The processes X_{t+1} and X_t are asymptotically dependent by construction and because of the standard Fréchet margins. Note that X_{t+h} and X_t have no noise variables Z in common for $h \geq 2$, so they are independent for such lags, as shown by the right plot of Figure 1 and the plots in Figure 2. Here we see that $\chi_h(u)$ behaves similarly for $h \geq 2$.

Figure 3 provides plots of $\chi(u)$ for the moving maximum for $a \in \{0.5, 0.1\}$, showing that the weakening dependence is reflected in a lower $\chi_h(u)$.

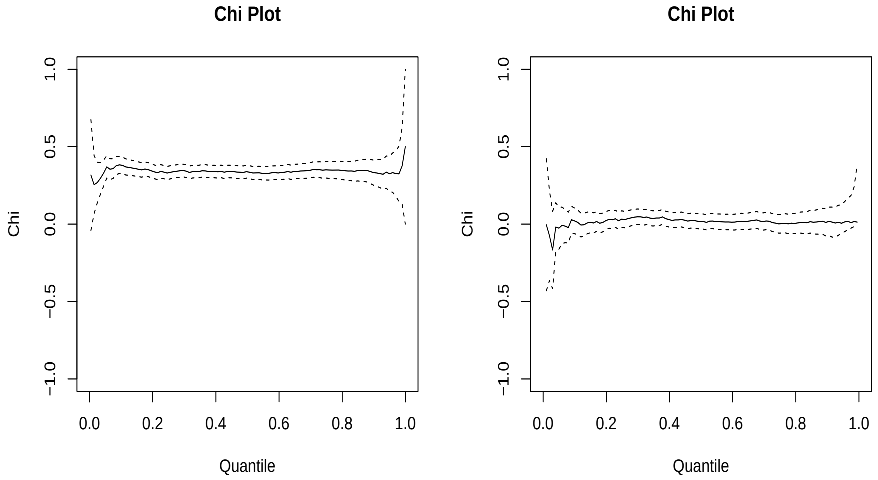

Figure 3: Plots of $\chi_h(u)$ for $a = 0.5$ and the lags $h = 1$ (left) and $h = 2$ (right) for the moving maximum process.

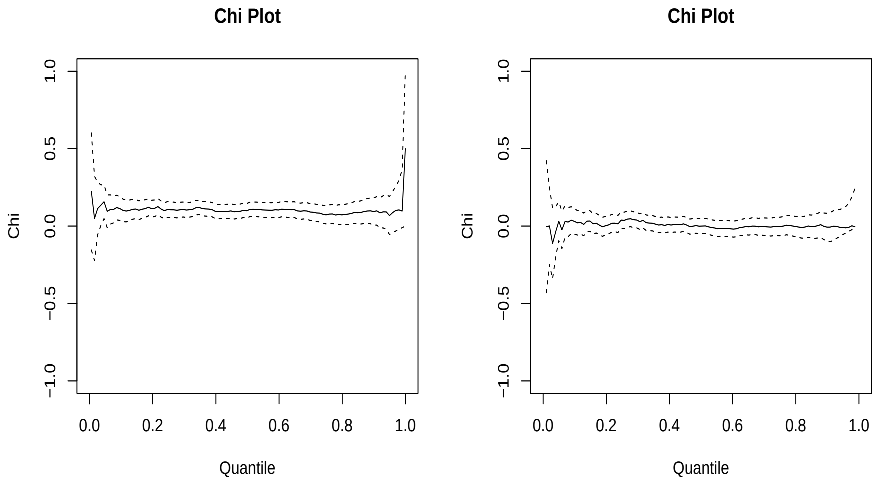

Figure 4: Plots of $\chi_h(u)$ for $a = 0.1$ and the lags $h = 1$ (left) and $h = 2$ (right) for the moving maximum process.

(b) For the Gaussian autoregressive process, we use bi-linearity of the covariance operator to compute

$$\begin{aligned}\operatorname{cov}(X_j, X_{j+1}) &= \operatorname{cov}(X_j, \rho X_j + (1-\rho^2)^{1/2}\varepsilon_{j+1}) \\ &= \rho \operatorname{cov}(X_j, X_j) + \operatorname{cov}(X_j, (1-\rho^2)^{1/2}\varepsilon_{j+1}) \\ &= \rho \operatorname{var}(X_j),\end{aligned}$$

since X_j is independent of ε_{j+1}. As $\operatorname{var}(X_j) = 1$, we find $\operatorname{corr}(X_j, X_{j+1}) = \rho$.

You were not asked to find the corresponding result for general h, but to do so write $\tilde{\varepsilon}_j = (1-\rho^2)^{1/2}\varepsilon_j$. Note that $\operatorname{cov}(X_{j+h}, X_j) = \operatorname{cov}(\rho X_{j+h-1} + \tilde{\varepsilon}_{j+h}, X_j) = \rho \operatorname{cov}(X_{j+h-1}, X_j)$ for $h \geq 1$, so one can start from $h = 1$ and use that $\operatorname{cov}(X_{j+h}, X_j) = \rho$ to show that $\operatorname{cov}(X_{j+2}, X_j) = \rho \operatorname{cov}(X_{j+1}, X_j)$. Proceeding similarly, and using that $\operatorname{var}(X_j) = 1$ gives $\operatorname{cov}(X_{j+h}, X_j) = \rho^h$. Applying a similar argument for $h < 0$ gives that $\operatorname{cov}(X_{j+h}, X_j) = \rho^{|h|}$.

In contrast to the moving maximum process with Fréchet margins, the Gaussian autoregressive process exhibits so-called asymptotic independence: while low levels of thresholds u show dependence,

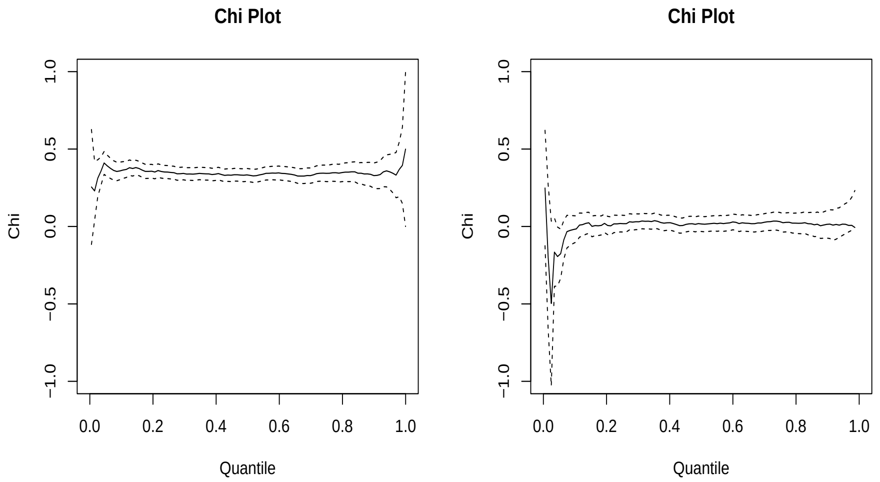

Figure 5: Plots of $\chi_h(u)$ for $a = 2$ and the lags $h = 1$ (left) and $h = 2$ (right) for the moving maximum process.

larger values of u lead to a decrease in χ, eventually giving $\lim_{u\to\infty} \chi(u) = 0$. Figure 6 shows plots of $\chi_h(u)$ for different lags h, indicating that convergence of $\chi(u)$ to zero is very slow.

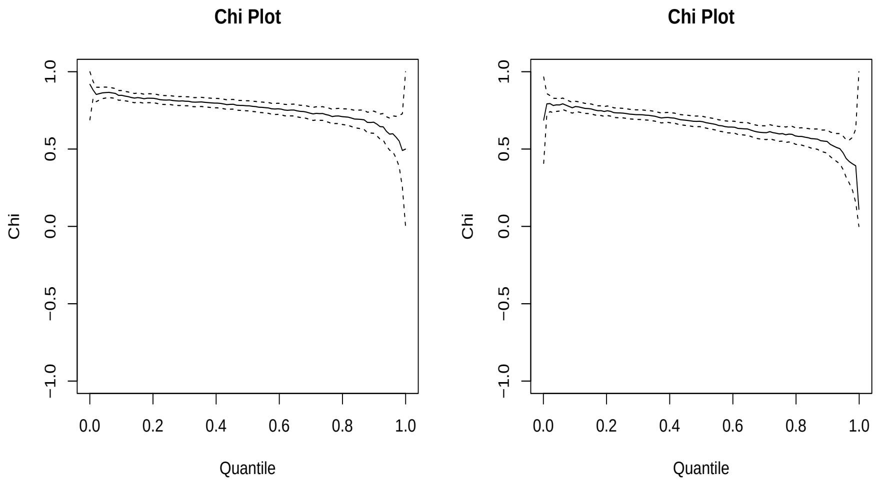

Figure 6: Plots of $\chi_h(u)$ for $a = 0.9$ and the lags $h = 1$ (left) and $h = 2$ (right) for the autoregressive process.

(c) We inspect the moving average process $X_{j+1} = \varepsilon_{j+1} + \rho\varepsilon_j$, and compute

$$\mathrm{var}(\varepsilon_{j+1} + \rho\varepsilon_j) = \mathrm{var}(\varepsilon_{j+1}) + \rho^2\mathrm{var}(\varepsilon_j) = 1 + \rho^2.$$

where we have used the independence of the ε_j's and that they have standard normal distributions. We then compute the covariance between X_{j+1} and X_j

$$\begin{aligned}\mathrm{cov}(\varepsilon_{j+1} + \rho\varepsilon_j, \varepsilon_j + \rho\varepsilon_{j-1}) &= \mathrm{cov}(\varepsilon_{j+1}, \varepsilon_j) + \mathrm{cov}(\varepsilon_{j+1}, \rho\varepsilon_{j-1}) + \mathrm{cov}(\rho\varepsilon_j, \varepsilon_j) + \mathrm{cov}(\rho\varepsilon_j, \rho\varepsilon_{j-1}) \\ &= \rho\mathrm{cov}(\varepsilon_j, \varepsilon_j) = \rho.\end{aligned}$$

Division by the variance leads to $\mathrm{corr}(X_{j+1}, X_j) = \rho/(1 + \rho^2)$.

For lags $h \geq 2$ the indices of the white noise ε involved in the computation of the covariance of X_{j+h} and X_j differ, and since the noise is independent, the processes X_{j+h} and X_j are independent for such h, implying that $\mathrm{cov}(X_{j+h}, X_j) = 0$. The dependence for the Gaussian process is illustrated by the left plot in Figure 7, exhibiting asymptotic independence similar to what we observed in

Figure 6. The plot on the right on Figure 7 shows independence similar to what we observed for lags $h \geq 2$ in (a).

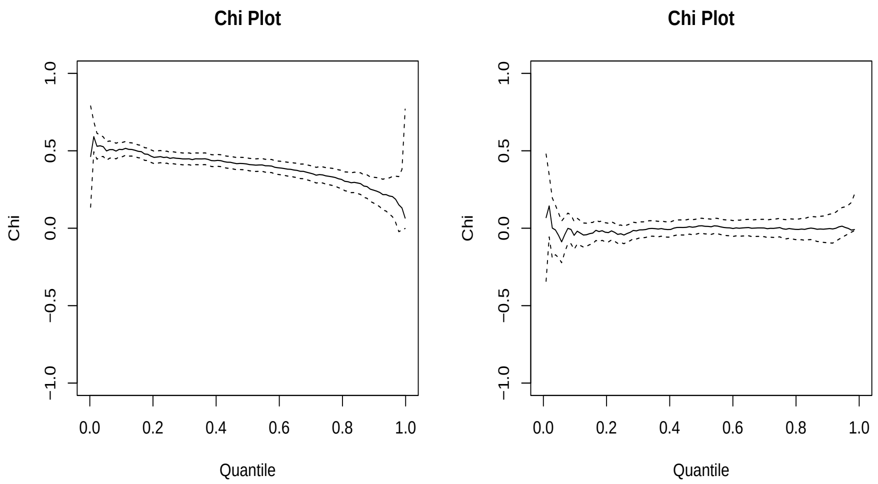

Figure 7: Plots of $\chi_h(u)$ for $a = 0.9$ and the lags $h = 1$ (left) and $h = 2$ (right) for the moving average process.

(d) To evaluate the impact of non-stationarity one must take into account both the availability of the data and the extent or frequency of the non-stationarity episodes relative to the observed data. For instance, if we observe daily data and non-stationary behaviour occurs on a yearly basis, we can expect its effect to be negligible. However, if we observe monthly or seasonal data, then we can expect the effect of such non-stationarity to be more significant.

Solution 2 Below we use the fact that the margins of the copula are uniform, and if $U \sim U(0,1)$, then $\mathrm{E}(U) = 1/2$ and $\mathrm{var}(U) = 1/12$.

(a) By the i.i.d. assumption, using that $\mathrm{P}(U_1 \geq V_1) = \mathrm{P}(U_1 \leq V_1) = 1/2$ we compute

$$\begin{aligned}
&\tau\mathrm{corr}(I\{U_1 > V_1\}, I\{U_2 > V_2\}) \\
&= \frac{\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}) - \mathrm{E}(I\{U_1 > V_1\})\mathrm{E}(I\{U_2 > V_2\})}{(\mathrm{E}(I\{U_1 > V_1\}) - \mathrm{E}(I\{U_1 > V_1\})^2)^{1/2}(\mathrm{E}(I\{U_2 > V_2\}) - \mathrm{E}(I\{U_2 > V_2\})^2)^{1/2}} \\
&= \frac{\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}) - (1/2)^2}{1/4} = 4\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}) - 1,
\end{aligned}$$

where $\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}) = \mathrm{E}(\mathrm{E}(I\{U_1 > V_1\}I\{U_2 > V_2\}|(U_1, U_2))) = E(C(U_1, U_2))$.

(b)

$$\rho = \mathrm{corr}(U_1, U_2) = \frac{\mathrm{E}(U_1U_2) - \mathrm{E}(U_1)\mathrm{E}(U_2)}{\mathrm{var}(U_1)^{1/2}\mathrm{var}(U_2)^{1/2}} = \frac{\mathrm{E}(U_1U_2) - \frac{1}{4}}{\frac{1}{12}} = 12\mathrm{E}(U_1U_2) - 3.$$

(c) When U_1 and U_2 are independent, we get a zero in the numerator in the second line of (a). Similarly in the first line in (b), $\mathrm{E}(U_1U_2) - 1/4 = \mathrm{E}(U_1)\mathrm{E}(U_2) - 1/4 = 0$.

Solution 3

(a) The copula is defined using $F_d^{-1}(u_d) = -1/\log u_d$ as

$$C(u_1, \ldots, u_D) = F\{F_1^{-1}(u_1), \ldots, F_1^{-1}(u_D)\} = \exp\{-V(-1/\log u_1, \ldots, -1/\log u_D)\}, \quad 0 < u_1, \ldots, < u_D.$$

(b) In terms of F we have $F^t(tz) = F(z)$, which when written as $F^t(z) = F(z/t)$ gives $C^t(u) = F^t(-1/\log u) = F(-1/t \log u) = F(-1/\log u^t) = C(u^t)$. Replacing u by $u^{1/t}$ gives the equation $C^t(u^{1/t}) = C(u)$, as required.

(c) It is easily checked that C_1 and C_2 are max-stable, but C_3 is not.