Solution 1 To see why the simulation algorithm works, note that if $E \sim \exp(1)$ and $Z = 1/E$, then

$$\mathrm{P}(Z \leq z) = \mathrm{P}(1/E \leq z) = \mathrm{P}(E \geq 1/z) = 1 - \{1 - \exp(-1/z)\} = \exp(-1/z), \quad z > 0,$$

which is the standard Fréchet CDF.

The first block of code gives Figure 1:

```
n <- 10000; a <- 1; i <- c(1:n)
z <- 1/rexp(n+1) # independent Frechet variables
x <- pmax(a*z[i],z[i+1])/(a+1) # moving maximum series
par(mfrow=c(1,2)) # two adjacent panels for figures
plot(i,x,log="y",pch=20, cex=.25) # should see clustering of high values,  but need log axes
qqplot(z,x,log="xy",cex=.25)
```

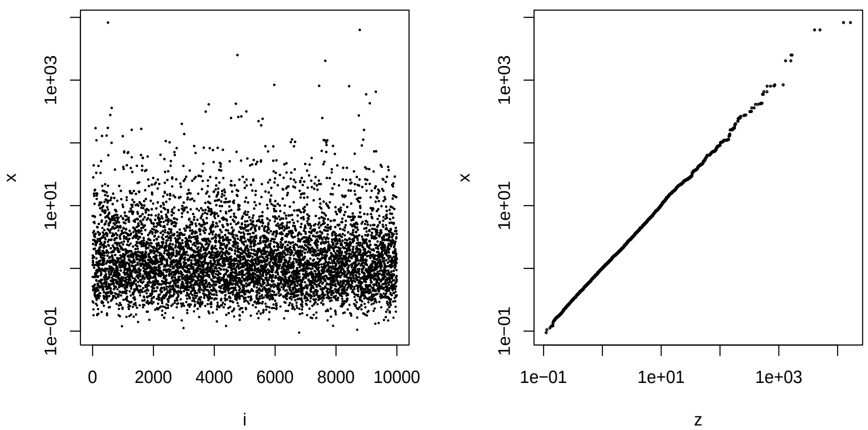

Figure 1: Plot of X (left) and QQplot of X and Z on log-scale (right).

The left-hand panel of Figure 1 may give the impression that the data are independent, but plotting the observations only in a shorter interval leads to a plot similar to that on Slide 146. The QQplot shows that the largest order statistics of X are tied, i.e., the first equals the second, the third equals the fourth and so on, because we take $a = 1$ in the formulation of X, whereas the largest order statistics of the independent process Z correspond to unique observations.

The code below results in Figure 2. Recall from Example 26 that $\theta = \max(1, a)/a + 1$, giving $\theta = 0.5$ if $a = 1$ and $\theta = 1$ if $a = 0$. We see that the estimate of θ approaches the true value as we increase the threshold.

```
tl <- quantile(z, probs = c(0.1,0.95))
exiplot(z,tl) #plots estimated theta between the limits given by tl
abline(h=1,col="red")
tl <- quantile(x, probs = c(0.1,0.95))
exiplot(x,tl)
abline(h=max(a,1)/(a+1),col="red")
```

We inspect the behaviour of $\widehat{\theta}_u$ for values of $a \in \{1, 1/3, 1/5, 1/7, 1/9\}$ and over different thresholds u using the code

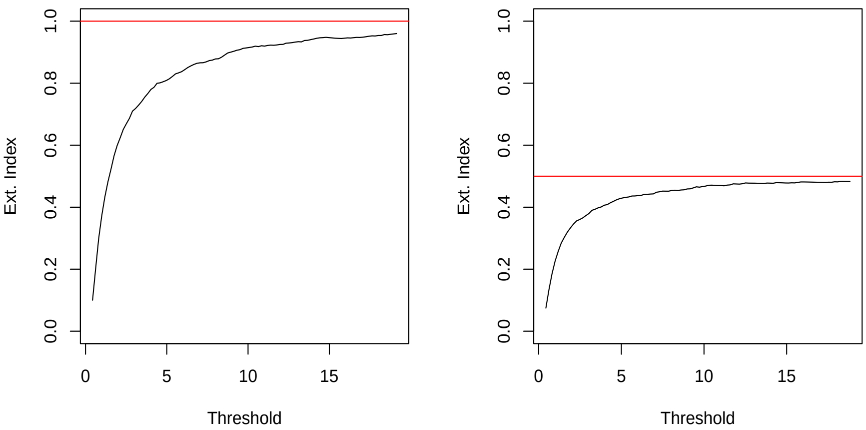

Figure 2: Estimated extremal indices for Z (left) and X (right).

```
par(mfrow=c(1,5))
for(a in seq(1,9, length=5)^-1){
n <- 10000; i <- c(1:n)
z <- 1/rexp(n+1) # independent Frechet variables
x <- pmax(a*z[i],z[i+1])/(a+1) # moving maximum series
tl <- quantile(x, probs = c(0.1,0.95))
exiplot(x,tl)
abline(h=max(a,1)/(a+1),col="red")
}
```

Figure 3 shows that the estimates $\widehat{\theta}_u$ behave reasonably well and that they give fairly good estimates of the extremal indices for sufficiently large values of the threshold u; these estimates are all biased downwards, however. The moving maximum model is very simplistic, so one must be cautious before reaching conclusions from the estimated $\widehat{\theta}$ in real data.

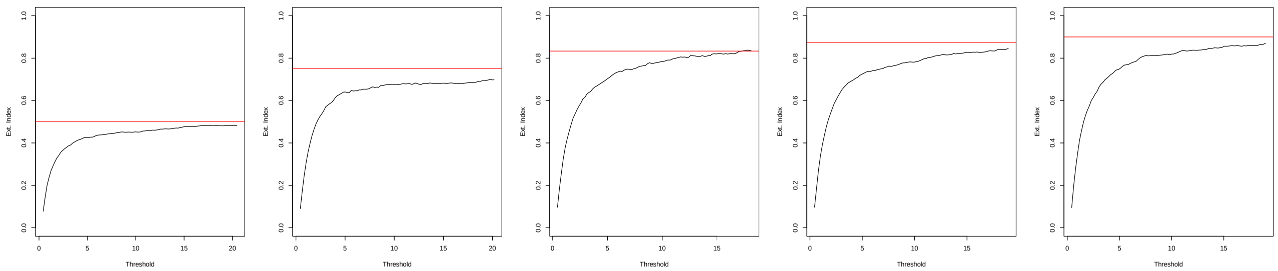

Figure 3: Left to right: Plot of $\widehat{\theta}_u$ for $a \in \{1, 1/3, 1/5, 1/7, 1/9\}$ as a function of the threshold u.

Solution 2

(a) We have $\mathrm{P}(S \le s) = 0$ for $s < 0$ and $\mathrm{P}(S \le 0) = 1 - \theta$, so $\mathrm{P}(S = 0) = 1 - \theta$. Hence

$$\mathrm{E}(S) = 0 \times (1-\theta) + \int_{0+}^{\infty} s \times \theta\lambda\theta e^{-\lambda\theta s}\,\mathrm{d}s = \theta(\lambda\theta)^{-1} = 1/\lambda,$$

and

$$\mathrm{E}(S \mid S > 0) = \int_{0+}^{\infty} s \times \theta\lambda\theta e^{-\lambda\theta s}\,\mathrm{d}s/\mathrm{P}(S > 0) = \theta(\lambda\theta)^{-1}/\theta = 1/(\lambda\theta),$$

so $\theta = \mathrm{E}(S)/\mathrm{E}(S \mid S > 0)$.

(b) If S^*/m has approximately the distribution of S, then $\mathrm{P}(S^* > s) = \mathrm{P}(S^*/m > s/m) \approx \mathrm{P}(S > s/m) = \theta \exp(-\lambda\theta s/m)$, for $s > 0$, and

$$\widehat{\theta} = \frac{n^{-1}\sum_{j=1}^n S_j^*/m}{n^{-1}\sum_{j=1}^n I(S_j^*/m > 1/m)S_j^*/m \div n^{-1}\sum_{j=1}^n I(S_j^*/m > 1/m)} \approx \mathrm{E}(S)/\mathrm{E}(S \mid S > 0)$$

for large n and m, so we might hope that the ratio is a reasonable estimator of θ. Proving this mathematically would require the asymptotic dependence between the S_j^* to be weak enough for the sums in $\widehat{\theta}$ to converge in probability to their expectations.

(c) The conditional distribution of $S \mid S > 0$ is exponential with parameter $\lambda\theta$, so $S - a \mid S > a \sim \exp(\lambda\theta)$ using the lack of memory of the exponential distribution. If you doubt this, note that for $x, a > 0$,

$$\mathrm{P}(S - a > x \mid S > a) = \mathrm{P}(S > a + x \mid S > a) = \frac{\mathrm{P}(S > a + x)}{\mathrm{P}(S > a)} = \frac{\exp\{-\lambda\theta(a + x)\}}{\exp(-\lambda\theta a)} = \exp(-\lambda\theta x).$$

For any $a > 0$, therefore, the positive values of $S_1^* - a, \ldots, S_n^* - a$, are a random sample from the exponential distribution with mean $\lambda\theta/m$. If $X_1, \ldots, X_K \overset{\text{iid}}{\sim} \exp(\lambda\theta/m)$ for known λ/m, then the log likelihood is

$$\ell(\theta) = K \log(\lambda\theta/m) - \lambda\theta/m \times \sum_{k=1}^K X_k, \quad \theta > 0,$$

so the MLE is $\widehat{\theta} = Km/(\lambda \sum_k X_k)$. In the present case K is replaced by $n_a = \sum_{j=1}^n I(S_j^* > a)$ and $\sum_{k=1}^K X_k$ is replaced by $\sum_{j=1}^n I(S_j^* > a)(S_j^* - a)$, so

$$\widehat{\theta} = \frac{m}{\lambda} \frac{\sum_{j=1}^n I(S_j^* > a)}{\sum_{j=1}^n I(S_j^* > a)(S_j^* - a)},$$

and if we replace λ/m by its estimator $n/\sum_{j=1}^n S_j^*$, we (almost) get the estimator in (b), with $a = 1$.

Solution 3

(a) As F, F_X, F_Y are distributions, they are monotone increasing, and thus $C(u, v)$ is also monotone increasing in each of its arguments. Moreover $C(u, v) \geq 0$ since $F(x, y) \geq 0$, and it is straightforward to show that $\lim_{u\to 0, v\to 0} C(u, v) = 0$ and $\lim_{u\to 1, v\to 1} C(u, v) = 1$, so $C(u, v)$ is a distribution function. Now, let (U, V) be distributed according to $C(u, v)$. Since $F_X(x) = F(x, \infty)$ and $F_Y(y) = F(\infty, y)$, we have

$$\begin{aligned}\mathrm{P}(U \leq u) &= \lim_{v\to 1} \mathrm{P}(U \leq u, V \leq v) = \lim_{v\to 1} C(u, v) = \lim_{v\to 1} F\left\{F_X^{-1}(u), F_Y^{-1}(v)\right\} \\ &= F\left\{F_X^{-1}(u), \lim_{v\to 1} F_Y^{-1}(v)\right\} = F\left\{F_X^{-1}(u), \infty\right\} = F_X\{F_X^{-1}(u)\} = u,\end{aligned}$$

and $\mathrm{P}(V \leq v) = v$ by symmetry: the margins of $C(u, v)$ are uniform.

(b) Recall that $\log(1 + a) \approx a$ as $a \to 0$, so $\log p = \log\{1 + (p - 1)\} \approx p - 1$ as $p \to 1$. Hence

$$\begin{aligned}\mathrm{P}\left\{Y > F_Y^{-1}(u) \mid X > F_X^{-1}(u)\right\} &= \frac{\mathrm{P}\left\{X > F_X^{-1}(u), Y > F_Y^{-1}(u)\right\}}{\mathrm{P}\left\{X > F_X^{-1}(u)\right\}} \\ &= \frac{1 - 2u + C(u, u)}{1 - u} = 2 - \frac{1 - C(u, u)}{1 - u} \\ &\approx 2 - \frac{\log C(u, u)}{\log u}, \quad u \to 1.\end{aligned}$$

The limit χ can be interpreted as a measure of extremal dependence. If $\chi = 0$, then the variables X and Y (and thus also U and V) are asymptotically independent. If $\chi > 0$, then the variables are asymptotically dependent. In practice, it often happens that dependence weakens at higher levels, casting doubt on the validity of asymptotically dependent models.

(c) Here $F_X(x) = \exp(-1/x)$ and $F_Y(y) = \exp(-1/y)$, so $F_X^{-1}(u) = -1/\log u$, $F_Y^{-1}(v) = -1/\log v$, and

$$C(u,v) = F\left\{F_X^{-1}(u), F_Y^{-1}(v)\right\} = \exp\left[-\left\{(-1/\log u)^{-1/\alpha} + (-1/\log v)^{-1/\alpha}\right\}^{\alpha}\right], \quad 0 < u, v < 1.$$

Thus,

$$\begin{aligned}\chi(u) &= 2 - \frac{\log C(u,u)}{\log u} = 2 - \frac{-\left\{(-1/\log u)^{-1/\alpha} + (-1/\log u)^{-1/\alpha}\right\}^{\alpha}}{\log u} \\ &= 2 - \frac{-\left\{2(-1/\log u)^{-1/\alpha}\right\}^{\alpha}}{\log u} = 2 - 2^{\alpha}\frac{-\left\{(-1/\log u)^{-1/\alpha}\right\}^{\alpha}}{\log u} = 2 - 2^{\alpha},\end{aligned}$$

and therefore $\chi = 2 - 2^{\alpha}$. When $\alpha = 1$ the variables are asymptotically independent (in fact, exactly independent) and $\chi = 0$, whereas $\chi \to 1$ as $\alpha \to 0$.

(d) The model in (c) has $\chi = \chi(u) = 2 - 2^{\alpha}$ for all $u \in [0,1]$, so $\chi(u) = 2 - 2^{0.3} \approx 0.77$ when $\alpha = 0.3$ and $\chi(u) = 0$ when $\alpha = 1$. The left- and right-hand graphs correspond to these models, so the middle one must correspond to the bivariate normal distribution. For the latter we see that there is dependence for all u but that the dependence reduces towards zero when $u \to 1$. In fact the bivariate normal model is asymptotically independent, ie., $\chi = 0$.