Problem 1

(a) Let $M_{m,j}$ denote the jth largest order statistic in a random sample of size m (so $M_{m,1}$ is the maximum), and suppose that the rescaled quantities $((M_{m,1}-b_m)/a_m, \ldots, (M_{m,r}-b_m)/a_m) \xrightarrow{D} (Y_1, \ldots, Y_r)$ as $m \to \infty$ for fixed r, where Y_1 has distribution function $\exp\{-\Lambda(y)\}$ and $Y_1 > \cdots > Y_r$. Explain why the cumulative distribution function of Y_2, conditional on $Y_1 = y_1$, must be $\exp\{\Lambda(y_1) - \Lambda(y_2)\}$ for $y_2 < y_1$, and deduce that the joint density of Y_1 and Y_2 must be

$$\{-\dot{\Lambda}(y_1)\}\{-\dot{\Lambda}(y_2)\} \exp\{-\Lambda(y_2)\}, \quad y_2 < y_1.$$

Hence show (e.g., by induction) that the joint density of $Y_1, \ldots, Y_r$ is $\exp\{-\Lambda(y_r)\} \prod_{j=1}^r \{-\dot{\Lambda}(y_j)\}$, with $y_r < \cdots < y_1$.

(b) Deduce that

$$f_{Y_r}(y_r) = \frac{\{-\dot{\Lambda}(y_r)\}\Lambda(y_r)^{r-1}}{(r-1)!} \exp\{-\Lambda(y_r)\}, \quad y_r \in \mathbb{R}.$$

(c) Hence show that the joint conditional density of $Y_1, \ldots, Y_r$ given that $Y_{r+1} = u$ is

$$r! \prod_{j=1}^{r} \frac{-\dot{\Lambda}(y_j)}{\Lambda(u)}, \quad y_1 > \cdots > y_r > u.$$

How is this related to the joint density of r independent variables, each having distribution function $H(y) = 1 - \Lambda(y)/\Lambda(u)$, for $y > u$? What is H?

Problem 2 Verify the formulae for x_p given on slide 102 of the notes.

Problem 3 Suppose that the GEV distribution with parameters η, τ and ξ has been fitted to data $y_1, \ldots, y_n$, and a profile log likelihood is required for the $1/p$-year return level x_p.

(a) Show that $x_p = \eta + \tau a_p(\xi)$, where for $p \in (0, 1)$ we set $a_p(\xi) = [\{-\log(1-p)\}^{-\xi} - 1]/\xi$.

(b) The required profile log likelihood $\ell_{\mathrm{p}}(x_p)$ is defined as $\max_{\tau,\xi} \ell^*(x_p, \tau, \xi)$, where ℓ^* is the log likelihood function parametrised in terms of x_p, τ and ξ. Show that

$$\ell_{\mathrm{p}}(x_p) = \max_{\tau,\xi} \ell\{x_p - \tau a_p(\xi), \tau, \xi\},$$

where ℓ is the log likelihood parametrized in terms of η, τ and ξ, and hence explain how you would compute $\ell_{\mathrm{p}}(x_p)$.