Problem 1

(a) Show that the generalized extreme-value distribution

$$G(y;\eta,\tau,\xi) = \exp\left\{-\left(1+\xi\frac{y-\eta}{\tau}\right)_+^{-1/\xi}\right\}, \quad -\infty < \xi, \eta < \infty, \tau > 0,$$

where $a_+ = \max(a, 0)$, satisfies the max-stability relation $G(y;\eta,\tau,\xi)^T = G(y;\eta_T,\tau_T,\xi_T)$, for $T > 0$, and give the forms of η_T, τ_T and ξ_T.

(b) Verify that $\lim_{\xi\to 0} G(y;\eta,\tau,\xi) = \exp\left[-\exp\left\{-(y-\eta)/\tau\right\}\right]$, and give η_T and τ_T in this case.

(c) Discuss the limiting behaviour of η_T, τ_T and ξ_T as functions of T.

(d) Find the support of $Y \sim G$ in the three cases $\xi > 0$, $\xi = 0$ and $\xi < 0$.

Problem 2

(a) If $Y \sim \mathrm{GEV}(\eta,\tau,\xi)$, show that $Y \overset{\mathrm{D}}{=} \eta + \tau X$, where $X \sim \mathrm{GEV}(0,1,\xi)$. Hence find the mean and variance of Y in terms of those of X. Let $G_0 = \exp\{-\Lambda_0(x)\}$ denote the CDF of X.

(b) Compute the mean and variance for X by using the transformation $z = (s+1)\Lambda_0(x)$ to show that its probability-weighted moments, if finite, may be written as

$$\mathrm{E}\left\{X^r G_0(X)^s\right\} = \frac{1}{s+1}\int_0^\infty \xi^{-r}\left\{\left(\frac{z}{s+1}\right)^{-\xi} - 1\right\}^r e^{-z}\,\mathrm{d}z, \quad r, s = 0, 1, 2, \ldots.$$

Then set $s = 0$ and $r = 1, 2$ to obtain the mean and variance of $Y = \eta + \tau X$.

Reminder: The gamma function and its rth derivative are

$$\Gamma(a) = \int_0^\infty u^{a-1}e^{-u}\,\mathrm{d}u, \quad \Gamma^{(r)}(a) = \int_0^\infty (\log u)^r u^{a-1}e^{-u}\,\mathrm{d}u, \quad a > 0;$$

for our purposes they are undefined when $a \le 0$. Note that $\Gamma(1) = 1$ and $\Gamma(a+1) = a\Gamma(a)$.

Problem 3 If $X \sim \mathrm{GPD}(\xi,\sigma)$, then $\mathrm{P}(X > x) = (1+\xi x/\sigma)_+^{-1/\xi}$ for $x > 0$, $\sigma > 0$ and $\xi \in \mathbb{R}$.

(a) Verify that $\lim_{\xi\to 0} \mathrm{P}(X > x) = \exp(-x/\sigma)$ for $x > 0$.

(b) Find the support of X in the three cases $\xi > 0$, $\xi < 0$ and $\xi = 0$.

(c) Find $\mathrm{E}(X)$.

(d) Show that if $x > 0$ and $x+u$ lies in the support of X, then $\mathrm{P}(X > u+x \mid X > u) = (1+\xi x/\sigma_u)_+^{-1/\xi}$, where $\sigma_u = \sigma + \xi u$, and deduce that $\mathrm{E}(X - u \mid X > u) = (\sigma + \xi u)/(1-\xi)$.

Hint for (c): if X is a positive-valued random variable with a finite mean, then $\mathrm{E}(X) = \int_0^\infty \mathrm{P}(X > x)\,\mathrm{d}x$.