Problem 1 The joint distribution function of $Z_1, \ldots, Z_D$ is of the form

$$G(z_1, \ldots, z_D) = \exp\{-V(z_1, \ldots, z_D)\}, \quad (z_1, \ldots, z_D) \in \mathcal{E}^* = [0, \infty)^D - \{(0, \ldots, 0)\},$$

and has unit Fréchet marginal distributions, i.e., $\mathrm{P}(Z_d \leq z_d) = \exp(-1/z_d)$ for each $d \in \mathcal{D} = \{1, \ldots, D\}$.

(a) Show that $V(z, \infty, \ldots, \infty) = 1/z$ for every permutation of the arguments of V, and that the unit Fréchet distribution is max-stable, i.e., there exist $a_t > 0$ and $b_t \in \mathbb{R}$ such that $\mathrm{P}(Z \leq b_t + a_t z)^t = \mathrm{P}(Z \leq z)$ for every $z, t > 0$.

(b) If the joint distribution G^* of $Z_1, \ldots, Z_D$ is max-stable, i.e., $\{G^*(tz)\}^t = G^*(z)$ for all t and z, show that $V(z_1, \ldots, z_D) = tV(tz_1, \ldots, tz_D)$ for every $t > 0$ and $(z_1, \ldots, z_D) \in \mathcal{E}^*$; V is said to be *homogeneous of order* -1. Deduce that $V(z, \ldots, z) = V(1, \ldots, 1)/z = \theta_{\mathcal{D}}/z$, say, and hence show that $\mathrm{P}\{\max(Z_1, \ldots, Z_D) \leq z\} = \exp(-\theta_{\mathcal{D}}/z)$, a Fréchet distribution with parameter $\theta_{\mathcal{D}}$.

(c) Let $\theta = V(1, 1)$ when $D = 2$. Show that $\chi = \lim_{z\to\infty} \mathrm{P}(Z_2 > z \mid Z_1 > z) = 2 - \theta$. Find θ when $V(z_1, z_2) = 1/z_1 + 1/z_2$ and $V(z_1, z_2) = 1/\min(z_1, z_2)$, and hence interpret it in terms of the degree of dependence between Z_1 and Z_2.

Problem 2 Show that the distribution

$$F(z_1, z_2) = \exp\left[-\left\{z_1^{-1} + z_2^{-1} + (z_1 z_2)^{-1}\right\}\right], \quad z_1, z_2 > 0,$$

has unit Fréchet margins. Is it max-stable? If not, what is the limiting distribution of appropriately rescaled componentwise maxima from F? Does this V correspond to the measure of a Poisson process?

Problem 3

(a) The bivariate Hüsler–Reiss model depends on a parameter $\lambda > 0$ and has exponent function

$$V(z_1, z_2) = \frac{1}{z_1}\Phi\left\{\frac{\lambda}{2} + \lambda^{-1}\log\left(\frac{z_2}{z_1}\right)\right\} + \frac{1}{z_2}\Phi\left\{\frac{\lambda}{2} + \lambda^{-1}\log\left(\frac{z_1}{z_2}\right)\right\},$$

where Φ denotes the standard normal distribution function. Show that letting $\lambda \to \infty$ and 0 yields independence and total dependence. Find $\theta = V(1, 1)$ for this model. and give its range of values.

(b) The bivariate negative logistic distribution has exponent function

$$V(z_1, z_2) = 1/z_1 + 1/z_2 - (z_1^\alpha + z_2^\alpha)^{-1/\alpha}, \quad \alpha > 0.$$

Show that letting $\alpha \to \infty$ and 0 yields total dependence and independence. Find $\theta = V(1, 1)$ for this model and give its range of values.