Problem 1 In Example 26 we met the moving maximum process with standard Fréchet margins and later we simulated it using the code

```
n <- 10000; a <- 1; i <- c(1:n)   # we saw this before
z <- 1/rexp(n+1)   # independent Frechet variables
x <- pmax(a*z[i],z[i+1])/(a+1)  # moving maximum series
par(mfrow=c(1,2))  # two adjacent panels for figures
```

In this exercise we shall use $\chi(u)$ to see compare the asymptotic (serial) dependence in this series with those in Gaussian time series models. The use of a copula means that we can take any margins, so we can just take the raw series and compute $\chi(u)$ between X_j and the lagged series X_{j+h}. Let's make a function to do this, using the option `which=1` to ensure that only the chi-plot is shown:

```
library(evd)
chi.lag <- function( x, lag=0)  chiplot(cbind(x[1:(n-lag)],x[(1+lag):n]), which=1)
```

(a) First for the moving maximum process, at lags $h = 1$ and $h = 2$:

```
chi.lag( x, 1)
chi.lag( x, 2)
```

Explain the difference between these plots. What would you expect to see for other lags? Check. Try this with other values of a.

(b) The stationary Gaussian autoregressive process of order one is defined in terms of a white noise process $\{\varepsilon_j\} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$ by the relationship

$$X_{j+1} = \rho X_j + (1 - \rho^2)^{1/2}\varepsilon_{j+1}, \quad j \in \mathbb{Z}, \quad \text{where} \quad |\rho| \leq 1.$$

Given that $\text{var}(X_j) = 1$ for all j, show that $\text{corr}(X_j, X_{j+1}) = \rho$; in fact $\text{corr}(X_j, X_{j+h}) = \rho^{|h|}$. We use the standard function `arima.sim` to generate data:

```
rho <- 0.9   # try other values also, including rho =1
x <- arima.sim(list(ar=rho),n)
chi.lag(x, 1)
chi.lag(x, 2)
```

This process is asymptotically independent. Describe how the chi-plots differ from those for the moving maximum process.

(c) The stationary Gaussian moving average process of order one is defined by

$$X_{j+1} = \rho\varepsilon_j + \varepsilon_{j+1}, \quad j \in \mathbb{Z}.$$

Show that $\text{corr}(X_j, X_{j+1}) = \rho/(\rho^2 + 1)$ and that $X_j \perp\!\!\!\perp X_{j+h}$ for $|h| \geq 2$. We again use `arima.sim`:

```
rho <- 0.9   # try other values also, including rho = 1
x <- arima.sim(list(ma=rho),n)
chi.lag(x, 1)
chi.lag(x, 2)
```

Discuss the chi-plots in relation to those above. Explain.

(d) Under what circumstances will lack of stationarity have a big impact on such plots?

Problem 2 Let the continuous random variables (U_1, U_2) and (V_1, V_2) be independent and identically distributed with copula $C(u_1, u_2)$.

(a) Show that Kendall's tau may be written $\tau = \text{corr}\left\{I(U_1 > V_1), I(U_2 > V_2)\right\} = 4\text{E}\left\{C(U_1, U_2)\right\} - 1$.

(b) Spearman's $\rho = \text{corr}(U_1, U_2)$ is another measure of concordance. Show that $\rho = 12\text{E}(U_1 U_2) - 3$.

(c) Show that if U_1 and U_2 are independent, then $\tau = \rho = 0$.

Problem 3 Let $Z = (Z_1, \dots, Z_D)$ have joint distribution function

$$F(z_1, \dots, z_D) = \exp\left\{-V(z_1, \dots, z_D)\right\}, \quad z_1, \dots, z_D > 0,$$

where, writing $z = (z_1, \dots, z_D)$ for brevity, $tV(tz) = V(z)$ for all $z > 0$ and the Z_d have marginal unit Fréchet distributions.

(a) Show that the copula corresponding to F is

$$C(u_1, \dots, u_D) = \exp\left\{-V(-1/\log u_1, \dots, -1/\log u_D)\right\}, \quad 0 < u_1, \dots, < u_D.$$

(b) Writing $u = (u_1, \dots, u_D)$ for brevity, show that the max-stability of F leads to the property $C(u^{1/t})^t = C(u)$. Such a copula is said to be *max-stable*.

(c) If $u = (u_1, \dots, u_D)$, $0 < \alpha < 1$ and $\rho > 0$, are the following copulas max-stable:

$$C_1(u) = \exp\left[-\left\{\sum_{d=1}^{D}(-\log u_d)^{1/\alpha}\right\}^{\alpha}\right], \quad C_2(u) = \exp\left[\sum_{d=1}^{D}\log u_d - \left\{\sum_{d=1}^{D}(-\log u_d)^{-\rho}\right\}^{-1/\rho}\right],$$

$$C_3(u) = \exp\left\{\sum_{d=1}^{D}\log u_d - \prod_{d=1}^{D}(-\log u_d)\right\}?$$