Problem 1 A moving maximum process (Example 26) with standard Fréchet marginal distribution can be simulated using the code

```
n <- 10000; a <- 1; i <- c(1:n)
z <- 1/rexp(n+1)   # independent Frechet variables
x <- pmax(a*z[i],z[i+1])/(a+1)  # moving maximum series
par(mfrow=c(1,2))  # two adjacent panels for figures
plot(i,x,log="y",pch=20)  # should see clustering of high values, but need log axes
qqplot(z,x,log="xy") # compare the marginal distributions of x and z
```

Here `z` contains independent data for which $\theta = 1$, since there should be no clustering, whereas `x` has clustered data. To see the effect of the clustering, we estimate the extremal index θ over a range of quantiles for each of these vectors:

```
tl <- quantile(z, probs = c(0.1,0.95))
exiplot(z,tl)  # plots estimated theta between the limits given by tl
abline(h=1,col="red")
tl <- quantile(x, probs = c(0.1,0.95))
exiplot(x,tl)
abline(h=max(a,1)/(a+1),col="red")
```

The red lines show the true values of θ for very large n; recall that $\theta = \max(a, 1)/(a + 1)$ for the moving maximum process.

Discuss the difference between the two plots. Try with other values of a. Do you see what you expect?

Problem 2 The non-negative random variable S has cumulative distribution function

$$\mathrm{P}(S \le s) = I(s \ge 0)\{1 - \theta \exp(-\lambda\theta s)\}, \quad s \in \mathbb{R}, \quad \lambda > 0, 0 < \theta \le 1.$$

(a) Show that $\mathrm{P}(S = 0) = 1 - \theta$, find $\mathrm{E}(S)$ and $\mathrm{E}(S \mid S > 0)$, and use them to give an expression for θ.

(b) A sample $S_1^*, \ldots, S_n^*$ takes values in the positive integers, but the limiting distribution of $S_1^*/m, \ldots, S_n^*/m$ as $m \to \infty$ when $n \to \infty$ is that of S. Show that this implies that S^* has approximately the distribution above with λ replaced by λ/m, and deduce that

$$\widehat{\theta} = \frac{n^{-1}\sum_{j=1}^n S_j^*}{n^{-1}\sum_{j=1}^n I(S_j^* > 1)S_j^* \div n^{-1}\sum_{j=1}^n I(S_j^* > 1)}$$

might be a reasonable estimator for θ.

(c) If S follows the model above and a is a positive constant, find the distribution of $S - a$ conditional on $S > a$. Hence find a maximum likelihood estimator for θ based on a sample $S_1^* - a, \ldots, S_n^* - a$, supposing that λ is known. How is this related to the estimator in (b)?

Problem 3 Consider continuous random variables X and Y taking values throughout $\mathbb{R}_+$ with joint distribution function $F(x, y)$ and marginal distribution functions $F_X(x)$ and $F_Y(y)$.

(a) Show that $C(u, v) = F\left\{F_X^{-1}(u), F_Y^{-1}(v)\right\}$ $(0 < u, v < 1)$ is a bivariate distribution function with uniform marginal distributions.

(b) Define $\chi(u) = 2 - \log C(u, u)/\log u$ for $0 < u < 1$ and show that

$$\lim_{u\to 1} \mathrm{P}\left\{Y > F_Y^{-1}(u) \mid X > F_X^{-1}(u)\right\} = \lim_{u\to 1} \chi(u) =: \chi.$$

(c) Find $C(u, v)$ and hence obtain $\chi(u)$ and χ when $F(x, y)$ equals $\exp\left\{-(x^{-1/\alpha} + y^{-1/\alpha})^{\alpha}\right\}$, $(x, y > 0)$, and $0 < \alpha \leq 1$. Comment on the cases $\alpha = 1$ and $\alpha \to 0$.

(d) The graph below shows $\chi(u)$ and its 95% confidence limits for samples from: (i) the model in (c) with $\alpha = 0.3$; (ii) the model in (c) with $\alpha = 1$; and (iii) a bivariate normal distribution with correlation 0.5. Say which panel corresponds to which model, justifying your reasoning.