

Risk and Environmental Sustainability

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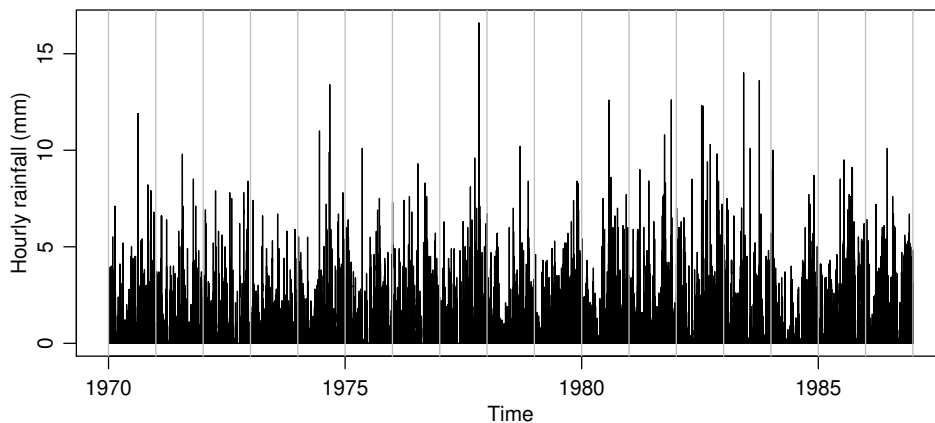
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Improved inferences

- The block maximum method can be inefficient if other data are available. Alternative methods include:
 - peaks over thresholds,
 - r -largest order statistics.
- Both are special cases of a **point process** representation, under which we use a Poisson process to approximate the occurrence of those values that exceed a (high) threshold.
- The Poisson process representation is also very powerful in more general settings, so we discuss it before returning to data analysis.

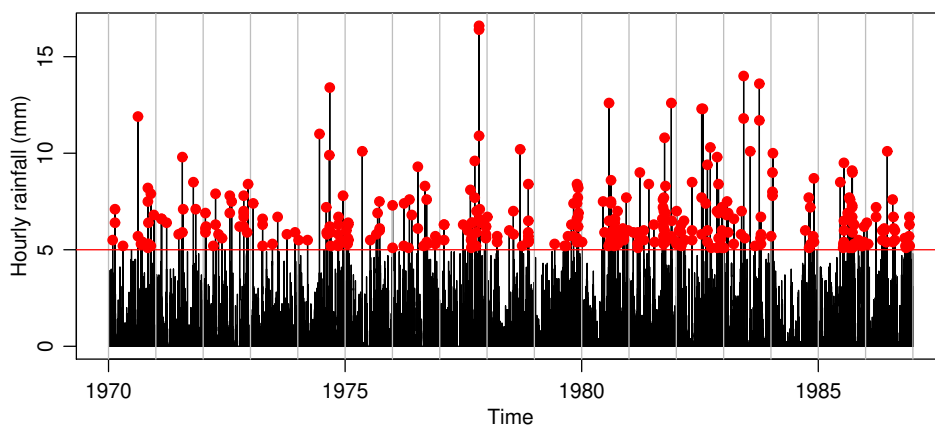
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Example: Hourly rainfall at Eskdalemuir, 1970–86



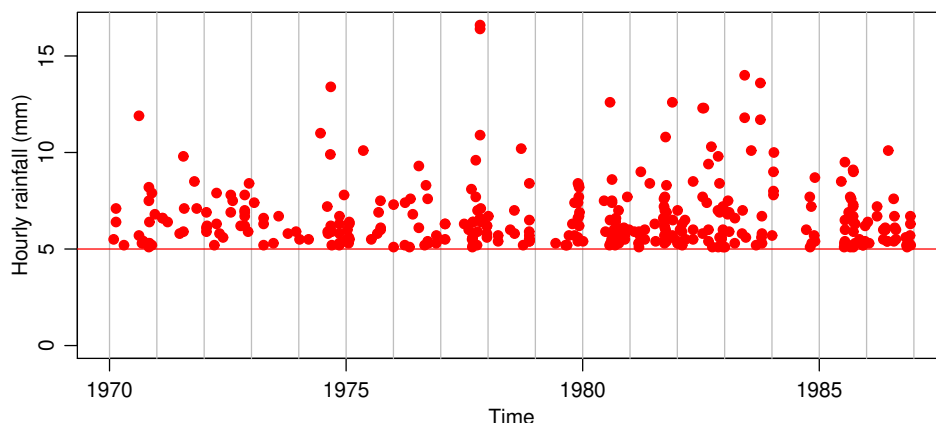
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Example: Eskdalemuir rainfall



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Example: Eskdalemuir rainfall



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Comments

- The fixed number of annual maxima has been replaced by a random number of exceedances over the threshold.
- We now retain more observations in the tail of the distribution.
- Dependence in the underlying series means that exceedances occur in clusters, which we may need to model.
- For now we suppose that the underlying series comprises independent identically distributed observations, whose maxima have a non-degenerate limit, after renormalisation.

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3.1 Point Processes

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Point process

- A **point process** is a stochastic model for a **point pattern** $\mathcal{P} = \{x_1, x_2, \dots\}$ lying in a **state space** \mathcal{E} . We also call a point an **event**.
- We visualise $\mathcal{E} \subset \mathbb{R}^2$, but \mathcal{E} might be more complex, e.g., $\mathcal{E} = \mathbb{R} \times \mathcal{C}$, where \mathcal{C} is a space of functions—then a ‘point’ would be $x = (u, f) \in \mathcal{E}$, with $u \in \mathbb{R}$ and $f \in \mathcal{C}$.
- The set \mathcal{E} must allow us to count how many points of \mathcal{P} lie in any suitable subset $\mathcal{A} \subset \mathcal{E}$, giving

$$N(\mathcal{A}) = |\mathcal{P} \cap \mathcal{A}| = \sum_x I(x \in \mathcal{P} \cap \mathcal{A}), \quad \mathcal{A} \subset \mathcal{E},$$

where $I(\cdot)$ is an indicator function.

- Two points cannot exactly coincide: \mathcal{P} must be **simple** (or **orderly**) — otherwise we would not know how many points there are.
- If you know about measures . . . the function $N(\mathcal{A})$ is
 - a **counting measure** on \mathcal{E} , since it counts the number of elements of \mathcal{P} in any (measurable) set \mathcal{A} ,
 - a **Radon measure** if $N(\mathcal{A}) < \infty$ for any \mathcal{A} compact (in a suitable topology on \mathcal{E}),
 - a **random measure** if the points \mathcal{P} arise at random, since then $N(\mathcal{A})$ is a random variable computed from the (random) \mathcal{P} .

Laplace transform

- If it exists, the **Laplace transform** of a scalar random variable X is defined as

$$E \{ \exp(-tX) \} = M_X(-t),$$

where M_X is the **moment-generating function (MGF)**. This is useful because

- there is a bijection between distributions and MGFs, i.e., if we recognise M_X , then we know the corresponding distribution;
- the **continuity theorem** tells us that if $\{X_n\}$, X have CDFs $\{F_n\}$, F for which the MGFs $M_n(t)$, $M(t)$ exist and there exists $a > 0$ such that

$$\lim_{n \rightarrow \infty} M_n(t) = M(t), \quad 0 \leq |t| < a,$$

then $X_n \xrightarrow{D} X$, i.e., X_n converges in distribution (weakly, in law) to X .

- Hence for large enough n we can approximate the distribution of X_n by that of X .
- On the next slide we will extend this to point processes, but first, a simple example:

Theorem 7 (Law of small numbers) *If $X_n \sim B(n, p_n)$ and $np_n \rightarrow \lambda > 0$ when $n \rightarrow \infty$, then the limiting distribution of X_n is $\text{Pois}(\lambda)$, i.e., $X_n \xrightarrow{D} X$, where $X \sim \text{Pois}(\lambda)$.*

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Note to Theorem 7

- The MGF of X is

$$M(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \lambda^x e^{-\lambda} / x! = e^{-\lambda} \sum_{x=0}^{\infty} (\lambda e^t)^x / x! = \exp \{ \lambda(e^t - 1) \}, \quad t \in \mathbb{R}.$$

- The MGF of X_n is

$$M_n(t) = E(e^{tX_n}) = \sum_{x=0}^n e^{tx} \binom{n}{x} p_n^x (1-p_n)^{n-x} = (1-p_n + p_n e^t)^n, \quad t \in \mathbb{R}.$$

Let $p_n = \lambda_n/n$, where $\lambda_n \rightarrow \lambda$, and note that as $n \rightarrow \infty$ and for any real t ,

$$(1-p_n + p_n e^t)^n = \left(1 + \frac{\lambda_n(e^t - 1)}{n} \right)^n \rightarrow \exp \{ \lambda(e^t - 1) \}.$$

- As $M_n(t) \rightarrow M(t)$ for all real t , the continuity theorem implies that $X_n \xrightarrow{D} X$.

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Laplace functional

- We specify properties of \mathcal{P} through the finite-dimensional distributions of $N(\cdot)$, i.e.,

$$P\{N(\mathcal{A}_1) = n_1, \dots, N(\mathcal{A}_k) = n_k\}, \quad n_1, \dots, n_k \in \{0, 1, 2, \dots\},$$

for all possible choices of sets $\mathcal{A}_1, \dots, \mathcal{A}_k$, and all $k = 0, 1, 2, \dots$

- An efficient way to do this is through the **Laplace functional**,

$$\mathcal{L}_{\mathcal{P}}(f) = E \left\{ \exp \left(- \int f d\mathcal{P} \right) \right\}, \quad \text{where} \quad \int f d\mathcal{P} = \int f(x) \mathcal{P}(dx) = \sum_{x \in \mathcal{P}} f(x),$$

for functions $f \geq 0$ that are positive only on a bounded set. If $f(x) = \sum_r t_r I(x \in \mathcal{A}_r)$, then $\mathcal{L}_{\mathcal{P}}(f)$ is the joint MGF for the $N(\mathcal{A}_r)$.

- Under mild conditions, there is
 - a bijection between point processes and Laplace functionals; and
 - the continuity theorem can be generalised.

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Convergence of point processes

Definition 8 A sequence of random variables $\{X_n\}$ with corresponding CDFs $\{F_n\}$ **converges weakly (or in distribution)** to a random variable X with CDF F if

$$\lim_{n \rightarrow \infty} F_n(x) = F(x) \text{ at every } x \text{ where } F \text{ is continuous.}$$

Definition 9 A sequence of point processes $\{\mathcal{P}_n\}$ with corresponding counts $\{N_n(\cdot)\}$ on \mathcal{E} **converges weakly (or in distribution)** to a point process \mathcal{P} with count $N(\cdot)$, written $\mathcal{P}_n \xrightarrow{D} \mathcal{P}$, if for all choices of k and all compact sets $\mathcal{A}_1, \dots, \mathcal{A}_k \subset \mathcal{E}$ such that

$$P\{N(\partial \mathcal{A}_j) = 0\} = 1, \quad j = 1, \dots, k,$$

where $\partial \mathcal{A}_j$ is the boundary of \mathcal{A}_j ,

$$\{N_n(\mathcal{A}_1), \dots, N_n(\mathcal{A}_k)\} \xrightarrow{D} \{N(\mathcal{A}_1), \dots, N(\mathcal{A}_k)\}, \quad n \rightarrow \infty.$$

Theorem 10 (No proof) The point processes $\mathcal{P}_1, \mathcal{P}_2, \dots$ converge weakly to the point process \mathcal{P} on \mathcal{E} if and only if the corresponding Laplace functionals converge for every continuous non-negative function f on \mathcal{E} with compact support, i.e., as $n \rightarrow \infty$,

$$\mathcal{L}_{\mathcal{P}_n}(f) = E \left\{ \exp \left(- \int f d\mathcal{P}_n \right) \right\} \rightarrow \mathcal{L}_{\mathcal{P}}(f) = E \left\{ \exp \left(- \int f d\mathcal{P} \right) \right\}.$$

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Kallenberg's theorem

- **Kallenberg's theorem** gives another way to establish the weak convergence of $\{\mathcal{P}_n\}$ to a simple process \mathcal{P} when $\mathcal{E} \subset \mathbb{R}$.
- For any $\mathcal{A} \subset \mathcal{E}$, let $N_n(\mathcal{A}) = |\mathcal{P}_n \cap \mathcal{A}|$. Then if
 - $\mathcal{B} \subset \mathcal{E}$ is any interval,
 - \mathcal{C} is any finite union of disjoint sub-intervals of \mathcal{E} ,
 and if

$$\mathbb{E}\{N_n(\mathcal{B})\} \rightarrow \mathbb{E}\{N(\mathcal{B})\}, \quad \mathbb{P}\{N_n(\mathcal{C}) = 0\} \rightarrow \mathbb{P}\{N(\mathcal{C}) = 0\}, \quad n \rightarrow \infty, \quad (8)$$

then \mathcal{P}_n converges weakly to \mathcal{P} .

- When $\mathcal{E} \subset \mathbb{R}^D$, the same result holds if intervals are replaced by **rectangles**,

$$(a, b] = \{x = (x_1, \dots, x_D) : a_d < x_d \leq b_d, d = 1, \dots, D\} \subset \mathcal{E},$$

where $a_d < b_d$ for each d .

- Thus weak convergence of point processes to a simple limiting process in \mathbb{R}^D entails establishing convergence of expected counts for rectangles and of the **void probabilities** of finite unions of rectangles.
- See Kingman (1993) *Poisson Processes* and Daley and Vere-Jones (2002, 2008), *An Introduction to the Theory of Point Processes*.

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3.2 Poisson Processes

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Poisson process

Definition 11 A **Poisson process** is a random countable subset \mathcal{P} of a state space \mathcal{E} such that

- the random variables $N(\mathcal{A}_1), \dots, N(\mathcal{A}_k)$ corresponding to any collection of disjoint subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$ of \mathcal{E} are independent; and
- for any $\mathcal{A} \subset \mathcal{E}$, $N(\mathcal{A})$ has the Poisson distribution with mean $\mu(\mathcal{A})$, where $0 \leq \mu(\mathcal{A}) \leq \infty$, and $\mu(\mathcal{A}) < \infty$ for compact \mathcal{A} .

Comments:

- if $\mathcal{A} = \bigcup_j \mathcal{A}_j$ is a countable union of disjoint sets, then $N(\mathcal{A}) = \sum_j N(\mathcal{A}_j)$, so $\mu(\mathcal{A}) = \sum_j \mu(\mathcal{A}_j)$, and μ is a measure; called the **mean measure** of \mathcal{P} ;
- μ must be **diffuse**, i.e., $\mu(\{x\}) = 0$ for every $x \in \mathcal{E}$;
- if $\mathcal{E} \subset \mathbb{R}^D$, $\mathcal{A} = [a_1, x_1] \times \dots \times [a_D, x_D]$, and if

$$\dot{\mu}(x_1, \dots, x_D) = \frac{\partial^D \mu(\mathcal{A})}{\partial x_1 \dots \partial x_D}$$

exists and is finite, then $\dot{\mu}$ is called the **intensity function** of \mathcal{P} ;

- if $\dot{\mu}(x) \equiv \dot{\mu}$, then \mathcal{P} is called **homogeneous**. Otherwise it is **inhomogeneous**.
- We simplify notation by replacing $\mu(\{a, b\})$ by $\mu(a, b]$, etc.

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Conditioning

Theorem 12 (Conditioning) Let \mathcal{P} be a Poisson process with mean measure μ , and suppose that $\mathcal{A} \subset \mathcal{E}$ is such that $0 < \mu(\mathcal{A}) < \infty$. Conditional on the event $N(\mathcal{A}) = n$, the n points of $\mathcal{P} \cap \mathcal{A}$ have the same distribution as n points generated independently at random in \mathcal{A} with measure $\mu_{\mathcal{A}}(\mathcal{B}) = \mu(\mathcal{B})/\mu(\mathcal{A})$, for $\mathcal{B} \subset \mathcal{A}$.

- If μ has intensity $\dot{\mu}(x)$, then we can generate points of \mathcal{P} in \mathcal{A} by
 - generating a value n of $N(\mathcal{A}) \sim \text{Poiss}\{\mu(\mathcal{A})\}$;
 - then generating $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \dot{\mu}(x)/\mu(\mathcal{A})$ for $x \in \mathcal{A}$.
- The process generated at the second step is a **binomial process**.

Lemma 13 The **Laplace functional** of a Poisson process \mathcal{P} on \mathcal{E} with mean measure μ is

$$\mathcal{L}_{\mathcal{P}}(f) = \exp \left[- \int_{\mathcal{E}} \{1 - e^{-f(x)}\} \mu(dx) \right].$$

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Note to Theorem 12

If we observe a Poisson process with intensity $\dot{\mu}(x)$ on the set \mathcal{A} , and there are points at $\{x_1, \dots, x_n\}$, then the corresponding probability element is

$$\exp\{-\mu(\mathcal{A})\} \times \prod_{j=1}^n \dot{\mu}(x_j), \quad \{x_1, \dots, x_n\} \subset \mathcal{A}.$$

Properties of the Poisson process imply that $N(\mathcal{A})$ has a Poisson distribution with mean $\mu(\mathcal{A})$, so the conditional density of the n points in \mathcal{A} , given that $N(\mathcal{A}) = n$, is the ratio

$$\frac{\exp\{-\mu(\mathcal{A})\} \times \prod_{j=1}^n \dot{\mu}(x_j)}{\mu(\mathcal{A})^n \exp\{-\mu(\mathcal{A})\}/n!} = n! \prod_{j=1}^n \left\{ \frac{\dot{\mu}(x_j)}{\mu(\mathcal{A})} \right\}, \quad \{x_1, \dots, x_n\} \subset \mathcal{A}.$$

Now consider the measure $\mu_{\mathcal{A}}(\mathcal{B}) = \mu(\mathcal{B})/\mu(\mathcal{A})$, for $\mathcal{B} \subset \mathcal{A}$, which is a probability measure on subsets of \mathcal{A} , because it is non-negative and $\mu_{\mathcal{A}}(\mathcal{A}) = 1$. The corresponding probability density is $\dot{\mu}(x)/\mu(\mathcal{A})$ ($x \in \mathcal{A}$), so the joint density for independent identically distributed variables X_1, \dots, X_n with distribution $\mu_{\mathcal{A}}$ is $\prod_{j=1}^n \{\dot{\mu}(x_j)/\mu(\mathcal{A})\}$, which is almost the conditional probability above. The additional factor $n!$ arises because the point process is unlabelled: the same density would arise for any of the $n!$ permutations of X_1, \dots, X_n that gave the outcome $\{x_1, \dots, x_n\}$.

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Note to Lemma 13

Let $f \geq 0$ have support only on a compact \mathcal{A} , so $\mu(\mathcal{A}) < \infty$. Conditional on $N(\mathcal{A}) = n$, $\int f(x)\mathcal{P}(dx) = \sum_{j=1}^n f(X_j)$, where $\{X_1, \dots, X_n\} \subset \mathcal{A}$ are independent with density $\dot{\mu}(x)/\mu(\mathcal{A})$. Thus

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ - \int f(x)\mathcal{P}(dx) \right\} \middle| N(\mathcal{A}) = n \right] &= \mathbb{E} \left[\exp \left\{ - \sum_{j=1}^n f(X_j) \right\} \middle| N(\mathcal{A}) = n \right] \\ &= \left\{ \int_{\mathcal{A}} e^{-f(x)} \mu(dx) / \mu(\mathcal{A}) \right\}^n. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ - \int f(x)\mathcal{P}(dx) \right\} \right] &= \sum_{n=0}^{\infty} \left\{ \int_{\mathcal{A}} e^{-f(x)} \mu(dx) / \mu(\mathcal{A}) \right\}^n \frac{\mu(\mathcal{A})^n}{n!} e^{-\mu(\mathcal{A})} \\ &= \exp \left[\int_{\mathcal{A}} e^{-f(x)} \mu(dx) - \mu(\mathcal{A}) \right] \\ &= \exp \left[- \int_{\mathcal{A}} \{1 - e^{-f(x)}\} \mu(dx) \right] \\ &= \exp \left[- \int_{\mathcal{E}} \{1 - e^{-f(x)}\} \mu(dx) \right], \end{aligned}$$

as required, since $1 - \exp\{-f(x)\} \equiv 0$ outside \mathcal{A} .

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Superposition and colouring

Theorem 14 (Superposition) *If $\mathcal{P}_1, \mathcal{P}_2$ are independent Poisson processes on \mathbb{R}^D with mean measures μ_1, μ_2 , then their union $\mathcal{P}_1 \cup \mathcal{P}_2$ is a Poisson process with mean measure $\mu_1 + \mu_2$.*

Theorem 14 extends to a countable number of Poisson processes.

Theorem 15 (Colouring) *Let \mathcal{P} be a Poisson process with intensity $\dot{\mu}(x)$. Colour a point of \mathcal{P} at x red with probability $\gamma(x)$; otherwise colour it green. Then the red and green sets of points \mathcal{P}_{red} and $\mathcal{P}_{\text{green}}$ are independent Poisson processes with intensity functions*

$$\dot{\mu}_{\text{red}}(x) = \dot{\mu}(x)\gamma(x), \quad \dot{\mu}_{\text{green}}(x) = \dot{\mu}(x)\{1 - \gamma(x)\}.$$

The colouring theorem is in some sense the inverse of the superposition theorem, and it too applies with a countable number of colours.

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Note to Theorem 14

- This looks easy using the Laplace functional for $\mathcal{P}_1 \cup \mathcal{P}_2$, which is

$$\mathcal{L}_{\mathcal{P}_1 \cup \mathcal{P}_2}(f) = \mathbb{E} \left[\exp \left\{ - \int f(x) (\mathcal{P}_1 \cup \mathcal{P}_2)(dx) \right\} \right].$$

Now

$$\int f(x) (\mathcal{P}_1 \cup \mathcal{P}_2)(dx) = \int f(x) \mathcal{P}_1(dx) + \int f(x) \mathcal{P}_2(dx),$$

and the two processes are independent, so

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ - \int f d(\mathcal{P}_1 \cup \mathcal{P}_2) \right\} \right] &= \mathbb{E} \left\{ \exp \left(- \int f d\mathcal{P}_1 \right) \right\} \times \mathbb{E} \left\{ \exp \left(- \int f d\mathcal{P}_2 \right) \right\} \\ &= \exp \left\{ - \int_{\mathcal{E}} (1 - e^{-f}) d\mu_1 \right\} \times \exp \left\{ - \int_{\mathcal{E}} (1 - e^{-f}) d\mu_2 \right\} \\ &= \exp \left\{ - \int_{\mathcal{E}} (1 - e^{-f}) d(\mu_1 + \mu_2) \right\}, \end{aligned}$$

which is the Laplace functional of a Poisson process with mean measure $\mu_1 + \mu_2$.

- The catch with the argument above is the assumption that points of \mathcal{P}_1 and \mathcal{P}_2 do not coincide, so that

$$\mathbb{P}(\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{A} = \emptyset) = 1$$

for any \mathcal{A} for which $\mu_1(\mathcal{A}), \mu_2(\mathcal{A})$ are both finite. This is intuitively obvious but takes a bit of measure-theoretic work to prove.

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Mapping

Theorem 16 (Mapping) Let \mathcal{P} be a Poisson process on \mathcal{E} with mean measure μ , and suppose that the function $g : \mathcal{E} \rightarrow \mathcal{E}^*$ maps \mathcal{E} into \mathcal{E}^* . Define

$$\mu^*(\mathcal{A}^*) = \mu\{g^{-1}(\mathcal{A}^*)\}, \quad \mathcal{A}^* \subset \mathcal{E}^*.$$

If

- (i) $\mu^*({x^*}) = \mu^*(x^*) = 0$ for every $x^* \in \mathcal{E}^*$, and
(ii) $\mu^*(\mathcal{A}^*) < \infty$ for any compact \mathcal{A}^* ,

then $\mathcal{P}^* = g(\mathcal{P})$ is a Poisson process on \mathcal{E}^* with mean measure μ^* .

Here

- (i) implies that g does not create atoms in \mathcal{E}^* ,
□ (ii) implies that no compact set $\mathcal{A}^* \subset \mathcal{E}^*$ has infinite measure,

which are both needed for \mathcal{P}^* to be Poisson.

Example 17 If \mathcal{P} is a homogeneous Poisson process of unit rate on $(0, \infty)$, and $g(x) = 1/x$, show that $g(\mathcal{P})$ is a Poisson process and find its intensity function. What if $g(x) = \lceil x \rceil$ or $g(x) = |\sin x|$?

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Note to Example 17

- The mean measure of \mathcal{P} is given by $\mu[a, b] = (b - a)$, for $0 < a < b < \infty$.
 - (i) The function g maps $(0, \infty)$ to $(0, \infty)$, and $g \equiv g^{-1}$, so $g^{-1}(x^*) = 1/x^*$ satisfies $\mu[1/x^*, 1/x^*] = (1/x^* - 1/x^*) = 0$ for any $0 < x^* < \infty$.
 - (ii) Any compact set \mathcal{A} of $(0, \infty)$ is a subset of a set $[a, b]$ for some $0 < a < b < \infty$, so

$$\mu^*(\mathcal{A}) = \mu(g^{-1}\mathcal{A}) = \int_{g^{-1}\mathcal{A}} \dot{\mu}(x) dx \leq \int_{g^{-1}[a,b]} 1 dx = \int_{1/b}^{1/a} dx = (1/a - 1/b) < \infty.$$

Hence $g(\mathcal{P})$ is indeed a Poisson process, and since $\mu[a, b] = (1/a - 1/b)$, its intensity function is $d\mu[a, b]/db = 1/b^2$, for $b > 0$.

A sketch shows what happens to the intensities of \mathcal{P} and $g(\mathcal{P})$.

- With $g(x) = \lceil x \rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x , then condition (i) fails whenever $x^* \in \mathbb{N}$, so the resulting process is not Poisson, as points of \mathcal{P}^* could be superposed on the positive integers and thus $N^*(\cdot)$ is not well-defined. Equivalently,

$$\mu^*({n}) = \mu[g^{-1}({n})] = \mu\{(n-1, n]\} = 1, \quad n \in \mathbb{N},$$

so μ^* has atoms on every positive integer and thus is not diffuse.

- With $g(x) = |\sin(x)|$ we have $\mathcal{E}^* = [0, 1]$, and it is easy to check that while condition (i) is satisfied, $\mu^*([a, b]) = \infty$ for any $0 < a < b < 1$, so condition (ii) fails; \mathcal{P}^* has an infinite number of points in any interval.

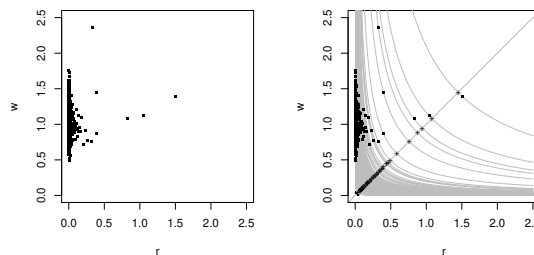
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Example

Example 18 Let \mathcal{P} be a Poisson process with $\mathcal{E} = \mathbb{R}_+^2$ with $x = (r, w)$ generated by

$$\mu\{(r, \infty) \times (w, \infty)\} = \frac{1}{r} \times \{1 - F(w)\}, \quad r, w > 0,$$

where F is the CDF of a positive continuous random variable W with unit expectation. Show that $q = rw$ defines a Poisson process and find its intensity.



Left panel: first 1000 points (r, w) of a Poisson process sequentially generated on \mathbb{R}_+^2 . Right panel: mapping of the points shown in the left panel to $q = rw$, shown as + on the diagonal, with the mapping function shown by the curved grey lines.

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Note to Example 18

- In the picture $\mathcal{P} = \{(R_i, W_i) : i = 1, 2, \dots\}$, where $R_1 > R_2 > \dots > 0$ are generated sequentially by setting $R_i = (E_1 + \dots + E_i)^{-1}$, with $E_i \stackrel{\text{iid}}{\sim} \exp(1)$, and $W_i = \exp(\sigma\varepsilon_i - \sigma^2/2)$, where $\varepsilon_i \stackrel{\text{iid}}{\sim} N(0, 1)$, independent of the E_i ; note that $E(W_i) = 1$. The first 1000 points of a realisation of such a process are shown in the left-hand panel of the figure; the full realisation would have an infinity of points at the left-hand edge of the panel, because $\mu\{(r, \infty) \times (0, \infty)\} = 1/r \rightarrow \infty$ as $r \rightarrow 0$.

- In the general case the mean measure has an *intensity function* $\dot{\mu}$ given by its derivative at the upper right corner of a rectangle $(r', r) \times (w', w)$, i.e.,

$$\begin{aligned} \dot{\mu}(r, w) &= \frac{\partial^2 \mu\{(r', r) \times (w', w)\}}{\partial r \partial w} \\ &= \frac{\partial^2}{\partial r \partial w} \{\mu(r', w') - \mu(r, w') - \mu(r', w) + \mu(r, w)\} \end{aligned} \quad (9)$$

$$= \frac{1}{r^2} \times f(w), \quad r, w > 0, \quad (10)$$

where we have written $\mu(r, w) = \mu\{(r, \infty) \times (w, \infty)\}$ and so forth, and f denotes the density function corresponding to F .

- Let $g(r, w) = rw$, corresponding to setting $Q_i = R_i W_i$, which amounts to collapsing the points shown in the left-hand panel onto the diagonal line shown in the right-hand panel. For any $q > 0$, $\mu^*(q) = \mu\{(r, q/r) : r > 0\} = 0$ because μ has a density with respect to Lebesgue measure and the set $\{(r, q/r) : r > 0\}$ has Lebesgue measure zero, so this transformation does not create atoms. We can check the second property of μ^* once it is calculated. Note that $Q = RW > q$ if and only if $R > q/W$, and that $\mathcal{A}_q = \{(r, w) : rw > q\}$ has measure

$$\begin{aligned} \mu^*(q) = \mu(\mathcal{A}_q) &= \int_0^\infty f(w) \int_{r=q/w}^\infty \frac{1}{r^2} dr dw \\ &= \int_0^\infty f(w) \left[-\frac{1}{r} \right]_{q/w}^\infty dw \\ &= \int_0^\infty f(w) \frac{1}{q/w} dw \\ &= \frac{1}{q} E(W) = \frac{1}{q}, \quad q > 0. \end{aligned} \quad (11)$$

Hence $Q_i = R_i W_i$ is also Poisson, with the same mean measure as the R_i . This implies that the second property is also satisfied: any compact set \mathcal{A}^* is a subset of (q_1, q_2) for some $q_2 > q_1$, so

$$\mu^*(\mathcal{A}^*) \leq \mu^*(q_1, \infty) = \mu^*(q_2, \infty) = q_1^{-1} - q_2^{-1} < \infty.$$

- The restriction of \mathcal{P} to a subset \mathcal{E}' of \mathcal{E} clearly also follows a Poisson process, with mean measure $\mu'(\mathcal{A}) = \mu(\mathcal{E}' \cap \mathcal{A})$. For example, if we let $\mathcal{E} = (0, \infty)$, consider R_1, R_2, \dots and let $\mathcal{E}' = (z', \infty)$ for some $z' > 0$, then we retain only those points R_i exceeding z' . As $\mu(\mathcal{E}') = 1/z'$ is finite, these R_i can be generated by first simulating a Poisson variable N' with mean $1/z'$, and if $N' = n$, simulating n independent variables on the interval (z', ∞) with survivor function z'/z ; these Pareto variables have probability density function z'/z^2 ($z > z'$).

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Marking

Theorem 19 (Marking) Let \mathcal{P} be a Poisson process on \mathcal{E} with mean measure μ . Attach a random variable y_x , called the mark, to each point x of \mathcal{P} ; the distribution of $y_x \in \mathcal{Y}$ may depend on x but not on any other point of \mathcal{P} . Then the points (x, y_x) form a Poisson process \mathcal{P}^* in the product space $\mathcal{E} \times \mathcal{Y}$ with mean measure

$$\mu(\mathcal{C}) = \iint_{(x,y) \in \mathcal{C}} \nu_x(dy) \mu(dx), \quad \mathcal{C} \subset \mathcal{E} \times \mathcal{Y},$$

where $\nu_x(\cdot)$ is the conditional probability measure of y_x given x .

- This provides an approach to making new Poisson processes, by attaching random variables to existing processes, and (perhaps) then applying the mapping theorem.
- If y_x takes a countable number of values (\equiv colours), then the colouring theorem shows that the corresponding subsets of \mathcal{P} are independent Poisson processes.

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Note to Theorem 19

The Laplace functional of \mathcal{P}^* is

$$\mathbb{E} \left\{ \exp \left(- \int f d\mathcal{P}^* \right) \right\} = \mathbb{E}_{\mathcal{P}} \left[\mathbb{E} \left\{ \exp \left(- \int f d\mathcal{P}^* \right) \right\} \middle| \mathcal{P} \right]$$

and the inner expectation on the right-hand side is

$$\prod_{x \in \mathcal{P}} \int_{\mathcal{Y}} e^{-f(x,y)} \nu_x(dy) = \exp \left(- \int f^* d\mathcal{P} \right),$$

say, where

$$f^*(x) = - \log \int_{\mathcal{Y}} e^{-f(x,y)} \nu_x(dy).$$

Thus the Laplace functional of \mathcal{P}^* is that of the Poisson process \mathcal{P} with f replaced by f^* . But since

$$\begin{aligned} \int_{\mathcal{E}} \left\{ 1 - e^{-f^*(x)} \right\} \mu(dx) &= \int_{\mathcal{E}} \left\{ 1 - \int_{\mathcal{Y}} e^{-f(x,y)} \nu_x(dy) \right\} \mu(dx) \\ &= \int_{\mathcal{E}} \int_{\mathcal{Y}} \left\{ 1 - e^{-f(x,y)} \right\} \nu_x(dy) \mu(dx), \end{aligned}$$

the result is established.

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Basic result

Theorem 20 Let $X_1, \dots, X_{nt_0} \stackrel{iid}{\sim} F$ form t_0 blocks each of n observations, and suppose that sequences $\{a_n\} > 0$ and $\{b_n\}$ exist such that

$$P[\{\max(X_1, \dots, X_n) - b_n\}/a_n \leq x] \rightarrow G(x), \quad n \rightarrow \infty,$$

where G is non-degenerate. Then as $n \rightarrow \infty$ the point processes

$$\mathcal{P}_n = \{(j/(n+1), (X_j - b_n)/a_n) : j = 1, \dots, nt_0\}$$

on $\mathcal{E} = [0, t_0] \times \mathcal{E}_x$ converge in distribution to a Poisson process \mathcal{P} with mean measure

$$\mu\{(t', t) \times [x, \infty)\} = (t - t')\Lambda(x), \quad 0 \leq t' < t \leq t_0, \quad x \in \mathcal{E}_x = \{x' \in \mathbb{R} : \Lambda(x') < \infty\}, \quad (12)$$

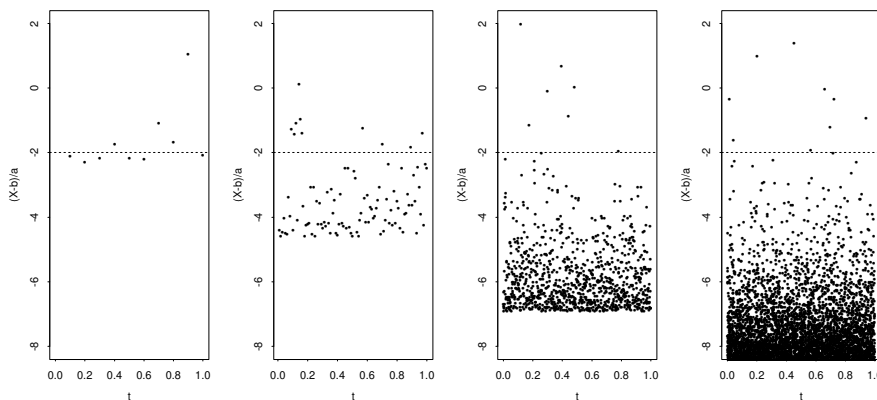
where

$$\Lambda(x) = \left(1 + \xi \frac{x - \eta}{\tau}\right)_+^{-1/\xi}$$

depends on parameters $\eta, \xi \in \mathbb{R}$ and $\tau > 0$ and $a_+ = \max(a, 0)$ for real a . The corresponding intensity function is

$$-\dot{\Lambda}(x) = \tau^{-1} \left(1 + \xi \frac{x - \eta}{\tau}\right)_+^{-1/\xi - 1} \geq 0.$$

Point process limit



□ Here $\mathcal{E} \subset \mathbb{R}^D$, so we only need Kallenberg's theorem: for $\mathcal{A} \subset \mathcal{E}$, let $N_n(\mathcal{A}) = |\mathcal{P}_n \cap \mathcal{A}|$. Then if $\mathcal{B} \subset \mathcal{E}$ is any rectangle, and \mathcal{C} is any finite union of disjoint rectangles of \mathcal{E} , and if

$$E\{N_n(\mathcal{B})\} \rightarrow E\{N(\mathcal{B})\}, \quad P\{N_n(\mathcal{C}) = 0\} \rightarrow P\{N(\mathcal{C}) = 0\}, \quad n \rightarrow \infty, \quad (13)$$

then $\mathcal{P}_n \xrightarrow{D} \mathcal{P}$ as $n \rightarrow \infty$.

Forms of $\Lambda(x)$

- $\Lambda(x)$ is decreasing, but has three distinct forms:

- when $\xi > 0$,

$$\Lambda(x) = \begin{cases} +\infty, & x \leq \eta - \tau/\xi, \\ (1 + \xi \frac{x-\eta}{\tau})_+^{-1/\xi}, & x > \eta - \tau/\xi, \end{cases}$$

which is finite only for $x > \eta - \tau/\xi$, so $\Lambda(\mathcal{A}) = +\infty$, giving infinite counts, for any set \mathcal{A} that goes below $\eta - \tau/\xi$;

- for $\xi = 0$ we take the limit when $\xi \rightarrow 0$, giving

$$\Lambda(x) = \exp\{-(x - \eta)/\tau\}, \quad x \in \mathbb{R},$$

which is finite for all x ;

- when $\xi < 0$,

$$\Lambda(x) = \begin{cases} (1 + \xi \frac{x-\eta}{\tau})_+^{-1/\xi}, & x < \eta - \tau/\xi, \\ 0, & x \geq \eta - \tau/\xi, \end{cases}$$

which is finite for all x .

- When $\xi \leq 0$ the limiting mass at $-\infty$ is infinite, so any set \mathcal{A} considered must have a finite lower bound.

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Implications: Maxima

- A rescaled block maximum $Y_n = \{\max(X_1, \dots, X_n) - b_n\}/a_n$ satisfies

$$\begin{aligned} P(Y_n \leq y) &= P[N_n\{(0, 1) \times [y, \infty)\} = 0] \\ &\rightarrow P[N\{(0, 1) \times [y, \infty)\} = 0] \quad n \rightarrow \infty, \\ &= \exp[-\mu\{(0, 1) \times [y, \infty)\}], \\ &= \exp\{-\Lambda(y)\}, \quad y \in \mathbb{R}, \end{aligned}$$

so a block maximum has a limiting **generalized extreme-value (GEV)** distribution,

$$G(y) = \exp\left\{-\left(1 + \xi \frac{y - \eta}{\tau}\right)_+^{-1/\xi}\right\}, \quad y \in \mathbb{R}. \quad (14)$$

- Let $Y_1 > \dots > Y_r$ denote the limits of the r largest rescaled order statistics in a block, so Y_1 is the maximum, etc. Then Y_2 is the largest of an infinite number of rescaled observations $(X_j - b_n)/a_n$, so its distribution is also G , but conditioned on $Y_2 < Y_1$. Hence

$$P(Y_2 \leq y_2 \mid Y_1 = y_1) = \exp\{\Lambda(y_1) - \Lambda(y_2)\}, \quad y_2 < y_1,$$

and it follows that the limiting joint density of the **r -largest order statistics** $Y_r < \dots < Y_1$ is

$$f(y_1, \dots, y_r) = \exp\{-\Lambda(y_r)\} \prod_{j=1}^r \{-\dot{\Lambda}(y_j)\}, \quad y_r < \dots < y_1. \quad (15)$$

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Implications: Threshold exceedances

- Consider the ‘forgetting’ mapping $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $g(t, x) = x$, giving the process of event sizes $\mathcal{P}^* = g(\mathcal{P})$ without their times. The mapping theorem (Theorem 16) implies that \mathcal{P}^* is Poisson with mean measure

$$\mu^*\{[x, \infty)\} = \mu[g^{-1}\{[x, \infty)\}] = \mu\{[0, t_0] \times [x, \infty)\} = t_0\{1 + \xi(x - \eta)/\tau\}_+^{-1/\xi}.$$

- For $y \in \mathbb{R}$, let $\mathcal{A}_y = [y, \infty)$ and let $N^*(\mathcal{A}_y) = |\mathcal{P}^* \cap \mathcal{A}_y|$.
- The conditional property (Theorem 12) implies that conditional on $N^*(\mathcal{A}_u) = n$, these n threshold exceedances have the same distribution as n points generated independently on \mathcal{A}_u with measure

$$\frac{\mu^*(\mathcal{A}_{u+x})}{\mu^*(\mathcal{A}_u)} = \frac{t_0\{1 + \xi(x - u - \eta)/\tau\}_+^{-1/\xi}}{t_0\{1 + \xi(u - \eta)/\tau\}_+^{-1/\xi}} = \left(1 + \xi \frac{x}{\sigma_u}\right)_+^{-1/\xi}, \quad x > 0,$$

where $\sigma_u = \tau + \xi(u - \eta)$.

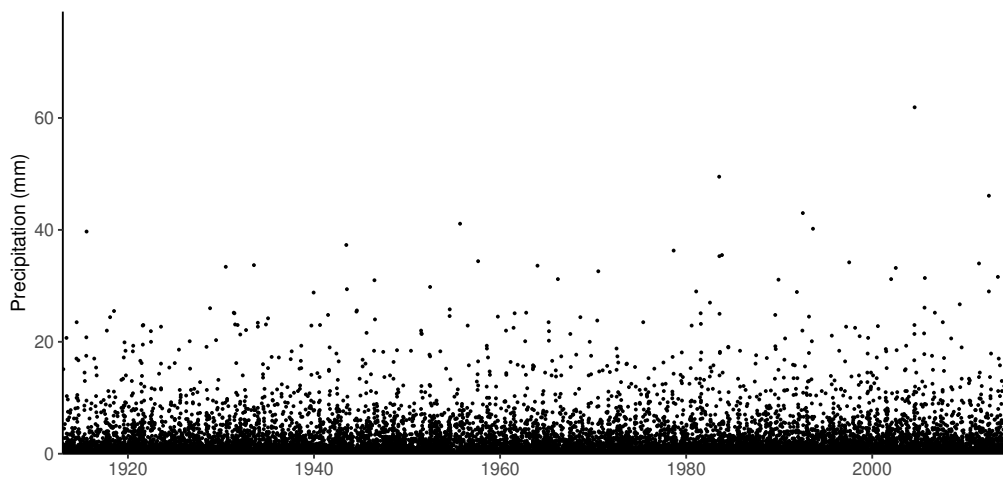
- Hence, provided that $\sigma_u > 0$, exceedances of the threshold u arise with rate $\Lambda(u)$ and are independent with the **generalized Pareto distribution (GPD)**

$$H(x) = 1 - \left(1 + \xi \frac{x}{\sigma_u}\right)_+^{-1/\xi}, \quad x > 0, \tag{16}$$

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Abisko daily rainfall data

- Daily precipitation in Abisko, in northern Sweden, 1913–2014. The largest value is 61.9 mm, but many values are zero and most of the positive values are quite small.



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Abisko Poisson process fit

```
(fit.pp <- fpot(abisko$precip, threshold=10, model="pp", npp=365.25,  
  start=list(loc=20,scale=6.5,shape=0.01))  
# needs initial values and number of points/block
```

Deviance: 2241.606

Threshold: 10

Number Above: 499

Proportion Above: 0.0134

Estimates

loc	scale	shape
19.79658	6.52110	0.07026

Standard Errors

loc	scale	shape
0.55597	0.37895	0.05088

Optimization Information

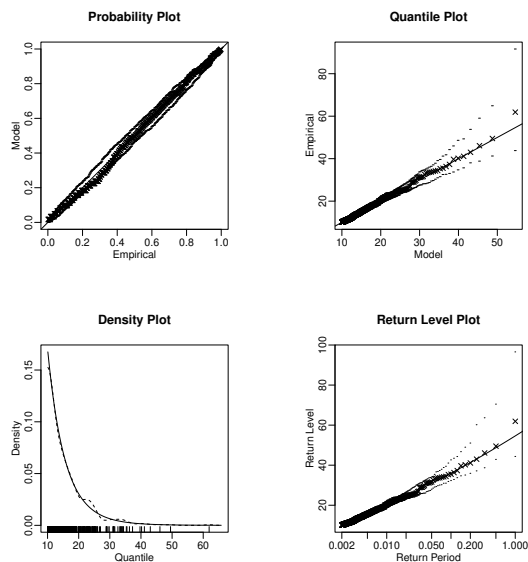
Convergence: successful

Function Evaluations: 20 ... Gradient Evaluations: 8

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Abisko Poisson process fit

Let's check the fit using `plot(fit.pp)`:



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Note 1 to Theorem 20

- If a limiting distribution G for rescaled maxima exists, then

$$\begin{aligned} P\{\{\max(X_1, \dots, X_n) - b_n\}/a_n \leq y\} &= P\{\max(X_1, \dots, X_n) \leq b_n + a_n y\} \\ &= F^n(b_n + a_n y) \\ &= \left[1 - \frac{n\{1 - F(b_n + a_n y)\}}{n}\right]^n. \end{aligned}$$

Hence a limiting function $\Lambda(y)$ must exist such that

$$\Lambda_n(y) = n\{1 - F(b_n + a_n y)\} \rightarrow \Lambda(y), \quad n \rightarrow \infty.$$

- Let $H(x) = -\log\{1 - F(x)\}$ denote the *cumulative hazard function* corresponding to F , and choose $b_n = b_n^*$ such that $H(b_n^*) = -\log n$, so that

$$\log \Lambda_n(y) = H(b_n + a_n y) - H(b_n).$$

- We suppose that F is continuous, places probability in an interval $[x_*, x^*]$, where either or both limits might be infinite, F is not defective (so there is no mass at x^*), that H is twice continuously differentiable with reciprocal hazard function $r(x) = 1/H'(x)$, and that $\lim_{x \rightarrow x^*} r'(x) = \xi$ is real and finite. These are sometimes called the *von Mises conditions*.

- Then

$$H(b_n + a_n y) - H(b_n) = a_n \int_0^y \frac{1}{r(b_n + a_n x)} dx = a_n \int_0^y \frac{1}{r(b_n) + a_n x r'\{b_n + s_n(x)\}} dx,$$

where $s_n(x)$ lies between zero and x . If we now choose $a_n = a_n^* = r(b_n^*)$, which is positive because $r(x) = \{1 - F(x)\}/f(x)$, we have

$$H(b_n^* + a_n^* y) - H(b_n^*) = \int_0^y \frac{1}{1 + x r'\{b_n^* + s_n(x)\}} dx = \int_0^y \frac{1}{1 + \xi_n x} g_n(x) dx,$$

where $\xi_n = r'(b_n^*)$ and $g_n(x) = (1 + \xi_n x)/\{1 + x r'\{b_n^* + s_n(x)\}\}$.

- The implicit function theorem implies that $s_n(x)$ is continuous in x and so is r' , so $g_n(x)$ is continuous in x , and one can check that $g_n(x) \rightarrow 1$ as $n \rightarrow \infty$. Hence in the interval where $1 + \xi_n x$ does not change sign, we can use a mean value theorem for integrals and choose y^* such that

$$H(b_n^* + a_n^* y) - H(b_n^*) = g_n(y^*) \int_0^y \frac{1}{1 + \xi_n x} dx = g_n(y^*) \times \xi_n^{-1} \log(1 + \xi_n y)_+,$$

where we add the $(\cdot)_+$ to remind us that the term in brackets must be positive. Now $\xi_n = r'(b_n^*) \rightarrow \xi$ and $0 < y^* < y$ as $n \rightarrow \infty$, so

$$\lim_{n \rightarrow \infty} H(b_n^* + a_n^* y) - H(b_n^*) = \xi^{-1} \log(1 + \xi y)_+ = \log \Lambda(y),$$

as required. This establishes sufficient conditions under which a maximum has limiting distribution $\exp\{-\Lambda(y)\}$, with $\eta = 0$ and $\tau = 1$. We need the more general case to allow for the fact that b_n and a_n are unknown in applications (because F is unknown).

note 1 of slide 114

Note 2 to Theorem 20

- To establish the Poisson convergence, define the binomial processes

$$\mathcal{P}_n = \{(j/(n+1), (X_j - b_n)/a_n) : j = 1, \dots, nt_0\}, \quad n = 1, 2, \dots,$$

and the corresponding count process $N_n(\cdot)$ on $\mathcal{E} = [0, t_0] \times \mathcal{E}_x$.

- Let $0 < t_1 < t_2 \leq t_0$ and $x_1 < x_2$ determine the rectangle $\mathcal{A} = (t_1, t_2] \times (x_1, x_2]$, let

$$\mu(\mathcal{A}) = (t_2 - t_1)\{\Lambda(x_1) - \Lambda(x_2)\}, \quad \mathcal{A} \subset \mathcal{E},$$

and let \mathcal{P} denote a Poisson process on \mathcal{E} with mean measure μ .

- We now check Kallenberg's conditions. If $[x]$ is the integer part of x , then

$$\begin{aligned} \mathbb{E}\{N_n(\mathcal{A})\} &= [(n+1)t_2 - (n+1)t_1] \times \mathbb{P}\{x_1 < (X_j - b_n)/a_n \leq x_2\} \\ &= \frac{[(n+1)(t_2 - t_1)]}{n} \times \Lambda_n(x_1, x_2) \\ &\rightarrow (t_2 - t_1)\Lambda(x_1, x_2) = \mu(\mathcal{A}), \quad n \rightarrow \infty, \end{aligned}$$

which verifies the first condition.

- For the second condition, let \mathcal{C} be a union of a finite number of disjoint rectangles of \mathcal{E} , and note that we can write $\mathcal{C} = \bigcup_{i=1}^k \mathcal{T}_i \times \bigcup_{l=1}^{L_i} \mathcal{X}_{i,l}$, where the $\mathcal{T}_i \subset [0, t_0]$ are disjoint intervals, and the intervals $\mathcal{X}_{i,l} \subset \mathbb{R}$ are disjoint for each i . Let $\mathcal{T}_1 = (t_1, t_2]$, let $\mathcal{X}_1 = \bigcup_{l=1}^{L_1} \mathcal{X}_{1,l}$ and $\mathcal{B}_1 = \mathcal{T}_1 \times \mathcal{X}_1$, and note that independence and identical distribution of the X_j gives

$$\begin{aligned} \mathbb{P}\{N_n(\mathcal{B}_1) = 0\} &= \mathbb{P}\{(X_1 - b_n)/a_n \notin \mathcal{X}_1\}^{[(n+1)(t_2 - t_1)]} \\ &= \left[\left\{ 1 - \frac{\Lambda_n(\mathcal{X}_1)}{n} \right\}^n \right]^{[(n+1)(t_2 - t_1)]/n} \\ &\rightarrow \exp\{-|\mathcal{T}_1| \Lambda(\mathcal{X}_1)\}, \quad n \rightarrow \infty, \\ &= \exp\{-\mu(\mathcal{T}_1 \times \mathcal{X}_1)\}. \end{aligned}$$

This applies for each \mathcal{T}_i , and the corresponding variables X_j are independent, so

$$\begin{aligned} \mathbb{P}\{N_n(\mathcal{C}) = 0\} &= \prod_{i=1}^k \mathbb{P}\{N_n(\mathcal{B}_i) = 0\} \\ &\rightarrow \prod_{i=1}^k \exp\{-\mu(\mathcal{T}_i \times \mathcal{X}_i)\}, \quad n \rightarrow \infty, \\ &= \exp\left\{-\sum_{i=1}^k \mu(\mathcal{B}_i)\right\} \\ &= \exp\{-\mu(\mathcal{C})\}, \end{aligned}$$

which establishes the second condition. Thus $\mathcal{P}_n \xrightarrow{D} \mathcal{P}$.