Solutions de la série n°11

Solution de l'exercice 1 :

1. $$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi}\,ax}{\varphi,\psi \vdash \varphi}\,aff}{\varphi \vdash \psi \to \varphi}\,\to i}{\vdash \varphi \to (\psi \to \varphi)}\,\to i$$

2. $$\dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi}\,ax}{\varphi \vdash \varphi \vee \psi}\,\vee ig}{\vdash \varphi \to (\varphi \vee \psi)}\,\to i$$

3. $$\dfrac{\dfrac{\dfrac{}{\varphi \wedge \psi \vdash \varphi \wedge \psi}\,ax}{\varphi \wedge \psi \vdash \varphi}\,\wedge eg}{\vdash (\varphi \wedge \psi) \to \varphi}\,\to i$$

4. $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\varphi \vdash \neg\varphi}\,ax \qquad \dfrac{}{\neg\neg\varphi \vdash \neg\neg\varphi}\,ax}{\neg\neg\varphi, \neg\varphi \vdash \bot}\,\neg e}{\neg\neg\varphi \vdash \varphi}\,\bot c}{\vdash \neg\neg\varphi \to \varphi}\,\to i \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi}\,ax \qquad \dfrac{}{\neg\varphi \vdash \neg\varphi}\,ax}{\varphi, \neg\varphi \vdash \bot}\,\neg e}{\varphi \vdash \neg\neg\varphi}\,\neg i}{\vdash \varphi \to \neg\neg\varphi}\,\to i}{\vdash \varphi \leftrightarrow \neg\neg\varphi}\,\wedge i$$

5. $$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\psi \vdash \neg\psi}\,ax \qquad \dfrac{}{\psi \vdash \psi}\,ax}{\psi, \neg\psi \vdash \bot}\,\neg e}{\psi \vdash \neg\neg\psi}\,\neg i \qquad \dfrac{\dfrac{\dfrac{}{\neg\varphi \vdash \neg\varphi}\,ax \qquad \dfrac{}{\varphi \vdash \varphi}\,ax}{\varphi, \neg\varphi \vdash \bot}\,\neg e}{\varphi \vdash \neg\neg\varphi}\,\neg i}{\varphi, \psi \vdash \neg\neg\varphi \wedge \neg\neg\psi}\,\wedge i$$

Solution de l'exercice 2 :

1. Symétrie de l'égalité :

avec $\phi(x) : x = x_1$
$t := x_1$ et $u := x_2$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash x_1 = x_1}\,= i \qquad \dfrac{}{x_1 = x_2 \vdash x_1 = x_2}\,ax}{x_1 = x_2 \vdash x_2 = x_1}\,= e}{\vdash x_1 = x_2 \to x_2 = x_1}\,\to i}{\vdash \forall x_2\ (x_1 = x_2 \to x_2 = x_1)}\,\forall i}{\vdash \forall x_1 \forall x_2\ (x_1 = x_2 \to x_2 = x_1)}\,\forall i$$

2. Transitivité de l'égalité :

avec $\phi(x) : x_1 = x$
$t := x_2$ et $u := x_3$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(x_1 = x_2 \wedge x_2 = x_3) \vdash (x_1 = x_2 \wedge x_2 = x_3)}\,ax}{(x_1 = x_2 \wedge x_2 = x_3) \vdash x_2 = x_3}\,\wedge ed \qquad \dfrac{\dfrac{}{(x_1 = x_2 \wedge x_2 = x_3) \vdash (x_1 = x_2 \wedge x_2 = x_3)}\,ax}{(x_1 = x_2 \wedge x_2 = x_3) \vdash x_1 = x_2}\,\wedge eg}{(x_1 = x_2 \wedge x_2 = x_3), (x_1 = x_2 \wedge x_2 = x_3) \vdash x_1 = x_3}\,= e}{(x_1 = x_2 \wedge x_2 = x_3) \vdash x_1 = x_3}\,ctr}{\vdash ((x_1 = x_2 \wedge x_2 = x_3) \to x_1 = x_3)}\,\to i}{\vdash \forall x_3\ ((x_1 = x_2 \wedge x_2 = x_3) \to x_1 = x_3)}\,\forall i}{\vdash \forall x_2 \forall x_3\, ((x_1 = x_2 \wedge x_2 = x_3) \to x_1 = x_3)}\,\forall i}{\vdash \forall x_1 \forall x_2 \forall x_3\, ((x_1 = x_2 \wedge x_2 = x_3) \to x_1 = x_3)}\,\forall i$$

Solution de l'exercice 3 :

1.

$$\dfrac{\dfrac{\dfrac{}{\neg(\varphi \lor \neg\varphi) \to (\varphi \lor \neg\varphi) \vdash \varphi \lor \neg\varphi}\,{\scriptstyle ax}}{\vdash (\neg(\varphi \lor \neg\varphi) \to \varphi \lor \neg\varphi) \to (\varphi \lor \neg\varphi)}\,{\scriptstyle \to i} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\varphi \lor \neg\varphi) \vdash \neg(\varphi \lor \neg\varphi)}\,{\scriptstyle ax} \qquad \dfrac{\dfrac{}{\varphi \vdash \varphi}\,{\scriptstyle ax}}{\varphi \vdash \varphi \lor \neg\varphi}\,{\scriptstyle \lor i}}{\neg(\varphi \lor \neg\varphi), \varphi \vdash \bot}\,{\scriptstyle \neg e}}{\neg(\varphi \lor \neg\varphi) \vdash \neg\varphi}\,{\scriptstyle \neg i}}{\neg(\varphi \lor \neg\varphi) \vdash \varphi \lor \neg\varphi}\,{\scriptstyle \lor id}}{\vdash \neg(\varphi \lor \neg\varphi) \to (\varphi \lor \neg\varphi)}\,{\scriptstyle \to i}}{\vdash \varphi \lor \neg\varphi}\,{\scriptstyle \to e}$$

2.

$$\dfrac{\dfrac{}{\vdash \varphi \lor \neg\varphi}\,{\scriptstyle ax} \qquad \dfrac{}{\varphi \vdash \varphi}\,{\scriptstyle ax} \qquad \dfrac{\dfrac{}{\neg\varphi \to \varphi \vdash \neg\varphi \to \varphi}\,{\scriptstyle ax} \qquad \dfrac{}{\neg\varphi \vdash \neg\varphi}\,{\scriptstyle ax}}{(\neg\varphi \to \varphi), \neg\varphi \vdash \varphi}\,{\scriptstyle \to e}}{(\neg\varphi \to \varphi) \vdash \varphi}\,{\scriptstyle \lor e}$$

3.

$$\dfrac{\dfrac{\dfrac{}{\neg\varphi \vdash \neg\varphi}\,{\scriptstyle ax} \qquad \dfrac{\dfrac{}{\neg\varphi \to \varphi \vdash \neg\varphi \to \varphi}\,{\scriptstyle ax} \qquad \dfrac{}{\neg\varphi \vdash \neg\varphi}\,{\scriptstyle ax}}{\neg\varphi \to \varphi, \neg\varphi \vdash \varphi}\,{\scriptstyle \to e}}{\neg\varphi \to \varphi, \neg\varphi \vdash \bot}\,{\scriptstyle \neg e + \text{ctr}}}{\neg\varphi \to \varphi \vdash \varphi}\,{\scriptstyle \bot c}$$

4.

$$\dfrac{\dfrac{\dfrac{}{\neg\varphi \to \varphi \vdash \varphi}\,{\scriptstyle ax}}{\vdash (\neg\varphi \to \varphi) \to \varphi}\,{\scriptstyle \to i} \qquad \dfrac{\dfrac{\Gamma, \neg\varphi \vdash \bot}{\Gamma, \neg\varphi \vdash \varphi}\,{\scriptstyle \bot e}}{\Gamma \vdash \neg\varphi \to \varphi}\,{\scriptstyle \to i}}{\Gamma \vdash \varphi}\,{\scriptstyle \to e}$$

5. Les points 1. et 2. nous montre que

$$\mathcal{P}_{\mathcal{R}_{int.+t.e.}} = \mathcal{P}_{\mathcal{R}_{int.+l.P.}}$$

Le point 3. nous montre l'inclusion suivante :

$$\mathcal{P}_{\mathcal{R}_{int.+l.P.}} \subseteq \mathcal{P}_{\mathcal{R}_{cl.}}$$

L'inclusion inverse découle du point 4. en observant que, dans toute déduction de la logique classique, si on remplace chaque occurrence de l'absurde classique par la dérivation obtenue au point 4., on obtient un arbre de preuve dans la logique $\mathcal{R}_{int.+l.P.}$