

Problem Sheet 8

November 3, 2025

Question 1

Graded exercise for group 2

Consider the 2D diffusion-convection problem :

$$-\epsilon \Delta u(x_1, x_2) + \frac{\partial u}{\partial x_1}(x_1, x_2) = 1 \quad (x_1, x_2) \in]0, 1[^2, \quad (1)$$

$$u(x_1, x_2) = 0 \quad (x_1, x_2) \in \partial]0, 1[^2. \quad (2)$$

implemented with order two centered finite differences schema.

- Fill the `diffconv2d.m` file to build the matrix of the linear system using only L (the number of points in each dimension), I (the identity $L \times L$ matrix) and E (the subdiagonal $L \times L$ matrix).
- Check the results when $\epsilon = 0.01$ and $L = 10, 20, 40, 80, 160$. Check that the number of iterations of the GMRES solver is $O(L)$.

Question 2

Let $\alpha > 0$, $T > 0$, $\beta \in C^0[0, 1]$, $y \in C^1[0, 1]$, such that for $0 \leq t \leq T$,

$$y'(t) + \alpha y(t) \leq \beta(t),$$

prove that

$$y(t) \leq y(0)e^{-\alpha t} + \int_0^t \beta(s)e^{-\alpha(t-s)} ds.$$

Hint : multiply by $e^{\alpha t}$.

Question 3

Let $T > 0$, $f : [0, 1] \times [0, T] \rightarrow \mathbb{R}$, $u_0 : [0, 1] \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$. We consider the nonlinear heat problem : find $u : [0, 1] \times [0, T] \rightarrow \mathbb{R}$ such that

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = f(x, t) + g(u(x, t)) & \text{in } [0, 1] \times [0, T], \\ u(x, 0) = u_0(x) & \text{in } [0, 1], \\ u(1, t) = u(0, t) = 0 & \text{in } [0, T]. \end{cases} \quad (3)$$

Assume $g(0) = 0$ and there exists L such that $\forall x, y \in \mathbb{R}$

$$|g(x) - g(y)| \leq L|x - y|.$$

— Show that

$$\frac{1}{2} \frac{d}{dt} \int_0^1 u(x, t)^2 dx + \int_0^1 \left(\frac{\partial u}{\partial x}(x, t) \right)^2 dx = \int_0^1 (f(x, t) + g(u(x, t))) u(x, t) dx.$$

— Show that

$$\frac{d}{dt} \int_0^1 u(x, t)^2 dx + (1 - 2L) \int_0^1 u(x, t)^2 dx \leq \int_0^1 f(x, t)^2 dx.$$

— Conclude that :

$$\int_0^1 u(x, t)^2 dx \leq \int_0^1 u_0(x)^2 dx e^{-(1-2L)t} + \int_0^t \left(\int_0^1 f(x, s)^2 dx \right) e^{-(1-2L)(t-s)} ds.$$

Question 4

Definition : The Kronecker product of two $m \times m$ matrices B and C is the $m^2 \times m^2$ matrix

$$B \otimes C = \begin{bmatrix} b_{11}C & \cdots & b_{1m}C \\ \vdots & & \vdots \\ b_{m1}C & \cdots & b_{mm}C \end{bmatrix}.$$

A Kronecker sum of two $m \times m$ matrices B and C is the $m^2 \times m^2$ matrix

$$(I \otimes B) + (C \otimes I)$$

where I is the $m \times m$ identity matrix.

Show the following theorem :

Theorem : Let B and C be two $m \times m$ matrices whose the eigenvalues are $\lambda_1, \dots, \lambda_m$ and μ_1, \dots, μ_m respectively. Then the eigenvalues of the Kronecker product are

$$\lambda_i \mu_j, \quad i, j = 1, \dots, m,$$

and the eigenvalues of the Kronecker sum are

$$\lambda_i + \mu_j, \quad i, j = 1, \dots, m.$$

Verify that $I_L \otimes A + A \otimes I_L$ has eigenvalues $4 - 2 \left(\cos \left(\frac{i\pi}{L+1} \right) + \cos \left(\frac{j\pi}{L+1} \right) \right)$ and $1 \leq i, j \leq L$, where A is the tridiagonal matrix $(-1, 2, -1)$ of dimension $L \times L$.