

# Problem Sheet 4

September 29, 2025

## Question 1

### Graded exercise for group 1

Let  $T > 0$ . Let  $y : [0, T] \rightarrow \mathbb{R}^d$  be the solution of the problem

$$y'(t) = f(t, y(t)), \quad 0 \leq t \leq T, \quad (1)$$

$$y(0) = y_0, \quad (2)$$

where  $y_0 \in \mathbb{R}^d$  and  $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a Lipschitz function of constant  $L$ .

Let  $N$  be a positive integer,  $h = T/N$ ,  $t_n = nh$ ,  $n = 0, 1, \dots, N$ . Consider the forward Euler scheme

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n), \quad n = 0, 1, \dots, N-1.$$

We define

$$y_h(t) = y_n \frac{t_{n+1} - t}{h} + y_{n+1} \frac{t - t_n}{h}, \quad t^n \leq t \leq t^{n+1}, n = 0, 1, 2, \dots, N-1.$$

(a) Prove that

$$y_h'(t) = f(t, y_h(t)) + \eta(t), \quad t_n \leq t \leq t_{n+1},$$

where  $\eta : [0, T] \rightarrow \mathbb{R}^d$  is computable if  $y_n, y_{n+1}$  are known.

(b) We set  $E(t) = y(t) - y_h(t)$ . Prove that

$$\frac{d}{dt} \|E(t)\|^2 \leq 4L \|E(t)\|^2 + \frac{\|\eta(t)\|^2}{2L}. \quad (3)$$

(c) Prove that

$$\|E(T)\|^2 \leq \int_0^T \frac{\|\eta(t)\|^2 e^{4L(T-t)}}{2L} dt. \quad (4)$$

Remark : the right hand side of (4) can be used to obtain an upper bound of the error  $E(T)$  once the  $y_n$  are computed,  $n = 1, \dots, N$ .

## Question 2

Implement the adaptive algorithm that was introduced in the lecture and apply it to ordinary differential equation given by

$$\begin{cases} \dot{y}(t) = -50(y(t) - \cos(t)), & 0 < t \leq T, \\ y(0) = 0.15, \end{cases} \quad (5)$$

with  $T = 7$ . Compute  $y_n$  with the Runge method and  $\hat{y}_n$  with the Forward Euler scheme.

## Question 3

Prove the following Lemma of the lecture. Let  $y$  be the solution to the ODE

$$\begin{cases} y'(t) = f(t, y(t)), & t_0 \leq t \leq T, \\ y(t_0) = y_0, \end{cases}$$

and  $y_n$  a RK method given by

$$\begin{aligned}y_{n+1} &= y_n + h_n \Phi(t_n, y_n, h_n), \\t_n &= t_{n-1} + h_n, \quad n = 0, 1, 2, \dots, N,\end{aligned}$$

where  $\Phi(t, z, h)$  is defined as in the lecture.

If  $f(t, z)$  satisfies a Lipschitz condition, that is there exists  $L > 0$  such that

$$\|f(t, z_1) - f(t, z_2)\| \leq L \|z_1 - z_2\|, \quad \forall (t, z_1), (t, z_2) \text{ in a neighbourhood of } (t, y(t)),$$

then there exists  $\Lambda > 0$  such that

$$\|\Phi(t, z_1, h) - \Phi(t, z_2, h)\| \leq \Lambda \|z_1 - z_2\|, \quad \forall t \leq T, 0 < h \leq h_{max}.$$