EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2025

Problem Sheet 2

September 15, 2025

Question 1

Let $y_0 \in \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be given, and let $y : \mathbb{R} \to \mathbb{R}$ be the solution of

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0. \end{cases} \tag{1}$$

Let $Y(t) = \begin{pmatrix} t \\ y(t) \end{pmatrix} \in \mathbb{R}^{n+1}$ such that

$$\begin{cases} Y'(t) = F(Y(t)), \\ Y(0) = \begin{pmatrix} t_0 \\ y_0 \end{pmatrix}, \end{cases} \tag{2}$$

where $F(Y(t)) = \begin{pmatrix} 1 \\ f(t, y(t)) \end{pmatrix}$.

Apply a general s stages explicit RK method to (2) and check that it corresponds to an s stages explicit RK method for (1) provided

$$c_i = \sum_{j=1}^{i-1} a_{ij}, i = 1, ..., s \text{ and } \sum_{i=1}^{s} b_i = 1.$$

Question 2

Graded exercise for group 2

Consider the ordinary differential equation given by

$$\begin{cases} \dot{y}(t) = \lambda y(t), & 0 < t \leqslant T, \\ y(0) = y_0, \end{cases} \tag{3}$$

with $\lambda < 0$. Let N be a positive integer, let $h = \frac{T}{N}$ be the time step and $t_n = nh$ where $n = 0, 1, ..., N$. Consider now an order 2 RK scheme with 2 stages to approximate (3). Let

$$p_2(x) = 1 + x + \frac{x^2}{2!}.$$

- Prove that $y_N = (p_2(\lambda h))^N y_0$.
- Prove that there exists $C > 0$ such that $\forall \lambda < 0, \forall h > 0$ such that $h \leqslant -\dfrac{2}{\lambda}$, $\forall T > 0$ and $\forall y_0 \in \mathbb{R}$,

$$|y(t_N) - y_N| \leqslant C|\lambda|^3 T h^2 |y_0|.$$