Problem Sheet 1

September 08, 2025

Question 1

Let d be a positive integer, $A \in \mathbb{R}^{d\times d}$ symmetric positive definite, $T > 0$, $\vec{u}_0 \in \mathbb{R}^d$, $\vec{f} : [0,T] \to \mathbb{R}^d$.
Let $\vec{u}(t) \in \mathbb{R}^d$ be the solution of the differential system :

$$\begin{aligned} \vec{u}'(t) + A\vec{u}(t) &= \vec{f}(t), \quad 0 < t \leqslant T, \\ \vec{u}(0) &= \vec{u}_0. \end{aligned} \tag{1}$$

- Prove that

  $$\|\vec{u}(T)\|^2 + \lambda_{\min} \int_0^T \|\vec{u}(t)\|^2 \; \mathrm{d}t \leqslant \frac{1}{\lambda_{\min}} \int_0^T \|\vec{f}(t)\|^2 \; \mathrm{d}t + \|\vec{u}_0\|^2,$$

  where $\lambda_{\min} > 0$ is the smallest eigenvalue of A and $\|\cdot\|$ denotes the Euclidean norm.

Let N be a positive integer, $h = \frac{T}{N}, t_n = nh, n = 0, 1, \ldots, N$. We compute $\vec{u}^{n+1} \in \mathbb{R}^d, n = 0, 1, \ldots, N-1$, with the Euler implicit scheme :

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{h} + A\vec{u}^{n+1} = \vec{f}(t_{n+1}). \tag{2}$$

- Assume that $\vec{u} \in \mathcal{C}^2([0,T], \mathbb{R}^d)$ and prove that

  $$\frac{\vec{u}(t_{n+1}) - \vec{u}(t_n)}{h} + A\vec{u}(t_{n+1}) = \vec{f}(t_{n+1}) + \vec{r}^{n+1}, \tag{3}$$

  with $\|\vec{r}^{n+1}\| \leqslant Ch$, where C depends only on u.
- Using the equality

  $$(a-b)a = \frac{1}{2}(a^2 - b^2 + (a-b)^2), \qquad a, b \in \mathbb{R},$$

  and setting $\vec{e}^{n+1} = \vec{u}(t_{n+1}) - \vec{u}^{n+1}$, prove that

  $$\frac{1}{h}\left(\|\vec{e}^{n+1}\|^2 - \|\vec{e}^n\|^2 + \|\vec{e}^{n+1} - \vec{e}^n\|^2\right) + \lambda_{\min}\|\vec{e}^{n+1}\|^2 \leqslant \frac{1}{\lambda_{\min}}\|\vec{r}^{n+1}\|^2,$$

  where $\lambda_{\min} > 0$ is the smallest eigenvalue of A.
- Conclude that $\|\vec{e}^N\|^2 \leqslant \dfrac{C^2}{\lambda_{\min}} Th^2$.

Question 2

Graded exercise for group 1

Consider the Euler explicit scheme to solve the differential system (1) :

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{h} + A\vec{u}^n = \vec{f}(t_n).$$

(a) Prove that

$$\frac{1}{2h}\left(\left\|\vec{e}^{n+1}\right\|^2 - \|\vec{e}^n\|^2 + \left\|\vec{e}^{n+1} - \vec{e}^n\right\|^2\right) + (\vec{e}^{n+1})^T A\vec{e}^{n+1} = (\vec{e}^{n+1})^T\vec{r}^n + (\vec{e}^{n+1})^T A(\vec{e}^{n+1} - \vec{e}^n),$$

where $\vec{r}^n$ is such that $\|\vec{r}^n\| \leqslant Ch$, where C depends only on u.

(b) Using Cauchy-Schwarz inequality for the scalar product $\vec{u}^T A\vec{v}$, $\vec{u}, \vec{v} \in \mathbb{R}^d$, prove that

$$(\vec{e}^{n+1})^T A(\vec{e}^{n+1} - \vec{e}^n) \leqslant \frac{1}{2}(\vec{e}^{n+1})^T A\vec{e}^{n+1} + \frac{1}{2}(\vec{e}^{n+1} - \vec{e}^n)^T A(\vec{e}^{n+1} - \vec{e}^n).$$

(c) Let $\lambda_{\min}$ (resp. $\lambda_{\max}$) be the smallest (resp. biggest) eigenvalue of A. Prove that

$$\frac{1}{2h}\left(\left\|\vec{e}^{n+1}\right\|^2 - \|\vec{e}^{n}\|^2 + \left\|\vec{e}^{n+1} - \vec{e}^{n}\right\|^2\right) + \frac{\lambda_{\min}}{2}\left\|\vec{e}^{n+1}\right\|^2 \leqslant \left\|\vec{e}^{n+1}\right\|\|\vec{r}^{n}\| + \frac{\lambda_{\max}}{2}\left\|\vec{e}^{n+1} - \vec{e}^{n}\right\|^2.$$

(d) Assume that $\frac{1}{2h} - \frac{\lambda_{\max}}{2} \geqslant 0$ and prove that

$$\left\|\vec{e}^{n+1}\right\|^2 \leqslant \|\vec{e}^{n}\|^2 + \frac{h}{\lambda_{\min}}\|\vec{r}^{n}\|^2,$$

so that

$$\left\|\vec{e}^{N}\right\|^2 \leqslant C^2\frac{h^2 T}{\lambda_{\min}}.$$

Question 3

Consider the differential system (Kepler)

$$\begin{aligned}\vec{x}''(t) &= \frac{-\vec{x}(t)}{\|\vec{x}(t)\|^3}, \quad 0 < t \leqslant T,\\ \vec{x}'(0) &= \vec{v}_0,\\ \vec{x}(0) &= \vec{x}_0,\end{aligned} \tag{4}$$

where $\vec{x}(t) = \begin{pmatrix} x_1(t)\\ x_2(t)\end{pmatrix}$ denotes the position at time t of a planet with respect to the sun and $\|\vec{x}(t)\|^3 = \left((x_1(t))^2 + (x_2(t))^2\right)^{3/2}$.

(a) Take the scalar product of (4) with $\vec{x}'(t)$ and prove that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{x}'(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\|\vec{x}(t)\|},$$

so that

$$\frac{1}{2}\|\vec{x}'(T)\|^2 - \frac{1}{\|\vec{x}(T)\|} = \frac{1}{2}\|\vec{v}_0\|^2 - \frac{1}{\|\vec{x}_0\|}.$$

(b) Introducing $\vec{x}'(t) = \vec{v}(t)$, write (4) as a differential system $\vec{u}'(t) = f(\vec{u}(t))$, with $\vec{u}(t) = \begin{pmatrix}\vec{x}(t)\\ \vec{v}(t)\end{pmatrix}$ and implement Euler explicit scheme. Check convergence when $\vec{x}_0 = \begin{pmatrix}1-c\\ 0\end{pmatrix}$, $\vec{v}_0 = \begin{pmatrix}0\\ \sqrt{\frac{1+c}{1-c}}\end{pmatrix}$, with $c = 0.6$, $T = 4\pi$.

Fill the following table containing the error for different time steps

h	$\|\vec{e}\|$ explicit

and provide a figure showing the results of the simulations.

Question 4

Ask chatgpt the following.

Given $a > 0$, write a javascript code to solve the ode $y' + ay = 0$ using the implicit Euler scheme and plot the results. Check stability and the convergence rate.

Question 5

Ask chatgpt the following.

Given A symmetric positive definite matrix, write a MATLAB code to solve the ode $y' + Ay = 0$ using the implicit Euler scheme and plot the results.
Check stability and the convergence rate.