Problem Sheet 9

November 10, 2025

Question 1

Let $T > 0$, $u_0 : [0,1] \to \mathbb{R}$ and $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, & 0 < x < 1,\ 0 < t \leqslant T; \\ u(0,t) = 0, & 0 < t \leqslant T; \\ u(1,t) = 0, & 0 < t \leqslant T; \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases} \tag{1}$$

We assume u_0 and u are smooth and we want to prove that if $u_0(x) \geqslant 0\ \forall x \in [0,1]$ then $u(x,t) \geqslant 0\ \forall (x,t) \in [0,1]\times[0,T]$. Let $u^-(x,t) := \max\{-u(x,t), 0\}$. Prove that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u^-(x,t))^2 dx + \int_0^1 (\frac{\partial u^-}{\partial x}(x,t))^2 dx = 0.$$

Hint : multiply the heat equation by $u^-(x,t)$ and integrate between $x = 0$ and $x = 1$.

Question 2

Consider the time dependent advection-diffusion problem : Given $T, \epsilon > 0$, find $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \epsilon\dfrac{\partial^2 u}{\partial x^2}(x,t) - \dfrac{\partial u}{\partial x}(x,t) = 1, & 0 < x < 1,\ t > 0; \\ u(0,t) = 0, & t > 0; \\ u(1,t) = 0, & t > 0; \\ u(x,0) = 0, & 0 < x < 1. \end{cases} \tag{2}$$

Let M, N be large integers, $h = \dfrac{1}{N+1}$, $\tau = \dfrac{T}{M}$, $x_i = ih$, $i = 0,1,\dots,N+1$, $t_n = n\tau$, $n = 0,1,\dots,M$.

(a) Proceed as in the lecture to write a difference scheme (write the PDE at time t_{n+1}, use a backward Finite Difference Formula for $\dfrac{\partial u}{\partial t}$ and a centered FDF for $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial^2 u}{\partial x^2}$).
Write the scheme as

$$(I + \tau A)\vec{u}^{n+1} = \tau\vec{1} + \vec{u}^n,\ n = 0,1,\dots,M-1,$$

where $\vec{u}^n$ has components u_i^n, $i = 1,\dots,N$, which are approximations of $u(x_i,t_n)$ and where A has to be defined.

Assume $h \leqslant 2\epsilon$.

(b) Prove that $||\vec{u}^{n+1}||_\infty \leqslant ||\vec{u}^n||_\infty + \tau$, so that $||\vec{u}^M||_\infty \leqslant T$.

(c) Prove that $u_i^n \geqslant 0$, $i = 1,\dots,N$, $n = 1,\dots,M$.

(d) Let $\vec{U}^n \in \mathbb{R}^N$ with components $u(x_i,t_n)$. Prove that, under reasonable assumptions on u,

$$(I + \tau A)\vec{U}^{n+1} = \tau\vec{1} + \vec{U}^n + \tau\vec{r}^{n+1},$$

where $||\vec{r}^{n+1}||_\infty \leqslant C(h^2 + \tau)$, with C independent of h and τ.

(e) Prove that $||\vec{U}^M - \vec{u}^M||_\infty \leqslant CT(h^2 + \tau)$.

Question 3

Graded Exercise for Group 3

Implement the scheme studied in Question 2. The Matlab file heat.m implements the heat equations, modify the file as required.

Check convergence : fix an $\epsilon = 0.01$ and play with the space and time step to provide a three digits approximation of $u(x = 0.9, t = 0.5)$.

What happens when :

— $T = 100,\ \epsilon = 0.01,\ N = 99,\ M = 100$?

— $T = 100,\ \epsilon = 0.01,\ N = 19,\ M = 100$?