

Problem Sheet 7

November 3, 2025

Question 1

Let $\epsilon > 0$ (small), we are looking for $u : [0, 1] \rightarrow \mathbb{R}$ such that

$$-\epsilon u''(x) + u'(x) = 1, \quad 0 < x < 1, \quad (1)$$

$$u(0) = u(1) = 0. \quad (2)$$

— Check that $u(x) = x - \frac{1-e^{-\frac{x}{\epsilon}}}{1-e^{-\frac{1}{\epsilon}}}$ and plot the solution for $\epsilon = 0.01$.

— Let N be a positive integer, $h = \frac{1}{N+1}$, $x_i = ih$, $i = 0, 1, \dots, N, N+1$. Consider the centered schema

$$-\epsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{u_{i+1} - u_{i-1}}{2h} = 1, \quad i = 1, \dots, N, \quad (3)$$

$$u_0 = u_{N+1} = 0. \quad (4)$$

Write the schema as a linear system $A\vec{u} = \vec{1}$ (define A).

— Let \vec{U} be the vector of exact solution $u(x_i)$, $i = 1, \dots, N$. Prove that $A\vec{U} = \vec{1} + \vec{r}$ where $|r_i| \leq Ch^2$, $i = 1, \dots, N$, with C independent of h .

— Prove that if $h \leq 2\epsilon$ then $A\vec{w} \geq \vec{1}$ where $w_i = x_i$, $i = 1, \dots, N$.

— Prove that if $h \leq 2\epsilon$ then $A\vec{z} \geq \vec{0}$ implies $\vec{z} \geq \vec{0}$ (use Question 2 of Sheet 6).

— Let $\vec{g} \in \mathbb{R}^N$ and \vec{v} such that $A\vec{v} = \vec{g}$. Then prove that $\|\vec{v}\|_\infty \leq \|\vec{g}\|_\infty$.

Prove that there exists $\tilde{C} > 0$ such that $\forall 0 < h < 2\epsilon$

$$\|\vec{U} - \vec{u}\|_\infty \leq Ch^2. \quad (5)$$

— Implement the problem.

(a) For $\epsilon = 0.01$ and $N = 9, 19, 39, 79, 159, 319, \dots$ check the convergence of the method.

(b) Take $N = 39$. What happens for $\epsilon = 0.1, 0.01, 0.001$?

Question 2

Graded exercise for group 1

Let $\epsilon > 0$ (small), we are looking for $u \in C^2([0, 1])$ such that

$$-\epsilon u''(x) + u'(x) = 1, \quad 0 < x < 1, \quad (6)$$

$$u(0) = u(1) = 0. \quad (7)$$

Prove that :

— $\forall v \in C^1([0, 1])$, $v(0) = v(1) = 0$ holds $\int_0^1 (v(x))^2 \leq \int_0^1 (v'(x))^2$.

— Prove $\left(\int_0^1 (u'(x))^2\right)^{1/2} \leq \frac{1}{\epsilon}$.

— Write the finite difference schema as $(\epsilon A + B)\vec{u} = \vec{1}$ where : A is symmetric positive definite and $B = -B^T$. Prove that,

$$\|\vec{u}\|_2 \leq \frac{\sqrt{N}}{\epsilon \lambda_1},$$

where λ_1 is the smallest eigenvalue of A .

