

Question 3

Consider the problem, given $f : [0, 1] \rightarrow \mathbb{R}$, find $u : [0, 1] \rightarrow \mathbb{R}$

$$\begin{cases} -u''(x) = f(x) & 0 < x < 1, \\ u(0) = 0 & u(1) = 0. \end{cases} \quad (3)$$

Let N be a positive integer, $h = \frac{1}{N+1}$, $x_i = ih$, $i = 0, 1, \dots, N+1$, and the schema (Numerov 1924)

$$\begin{cases} \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = \frac{f(x_{i-1}) + 10f(x_i) + f(x_{i+1}))}{12} & i = 1, \dots, N, \\ u_0 = 0 & u_{N+1} = 0. \end{cases} \quad (4)$$

Prove that $\forall f \in C^4[0, 1]$, $\exists C > 0$, $\forall 0 < h \leq 1$,

$$\max_{1 \leq i \leq N} |u(x_i) - u_i| \leq Ch^4. \quad (5)$$