Problem Sheet 5

October 6, 2025

Question 1

Consider a particle of mass m attached to a spring of stiffness k. The position of the particle is denoted $X(t)$. Newton's equation, together with initial conditions, writes :

$$\begin{cases} mX''(t) + kX(t) = 0, \quad 0 \leqslant t \leqslant T, \\ X(0) = X_0, \\ X'(0) = V_0. \end{cases} \tag{1}$$

1. Check that

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}m\big(X'(t)\big)^2 + \frac{1}{2}k\big(X(t)\big)^2\right) = 0.$$

2. Let N be the number of time steps, $h = \frac{T}{N}$ and $t_n = nh$ where $n = 0, 1, ..., N$. Let X_n be the approximation of $X(t_n)$ obtained using the Newmark's scheme :

$$m\frac{X_{n+1} - 2X_n + X_{n-1}}{h^2} + k\frac{X_{n+1} + 2X_n + X_{n-1}}{4} = 0, \ n = 1, \dots, N-1. \tag{2}$$

How can X_1 be computed ?
3. Prove that

$$\frac{1}{2}m\Big(\frac{X_{n+1} - X_n}{h}\Big)^2 + \frac{1}{2}k\Big(\frac{X_{n+1} + X_n}{2}\Big)^2 = \frac{1}{2}m\Big(\frac{X_n - X_{n-1}}{h}\Big)^2 + \frac{1}{2}k\Big(\frac{X_n + X_{n-1}}{2}\Big)^2.$$

Question 2

Let $f \in C^0([0,1])$, $a \in C^0([0,1])$, $a(x) > 0$ and $u \in C^2([0,1])$ be such that

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}(x)\right) = f(x), \qquad 0 < x < 1, \tag{3}$$

$$u(0) = u(1) = 0. \tag{4}$$

Let $N > 0$. $h = \frac{1}{N+1}$ and $x_i = ih$, $i = 0, 1, ..., N+1$. Let u_i be the approximation value of $u(x_i)$, $i = 0, 1, ..., N$. Consider the finite difference scheme

$$-\frac{1}{h}\left(a(x_{i+\frac{1}{2}})\frac{u_{i+1} - u_i}{h} - a(x_{i-\frac{1}{2}})\frac{u_i - u_{i-1}}{h}\right) = f(x_i), \qquad i = 0, 1, ..., N,$$

with $u_0 = 0 = u_{N+1}$.

- Check that A is symmetric positive definite.
- The matlab file diffin.m implements the method when $a(x) = 1$. Implement the scheme when $a(x) = 1 + x$ and $u(x) = x\frac{1-x}{2}$. Check numerically that : $max_{1 \leqslant i \leqslant N}|u(x_i) - u_i| = O(h^2)$

Question 3

Graded Exercise for Group 2

(a) Let $f \in C^0[0,1]$ such that

$$-u''(x) + u'(x) + u(x) = f(x) \quad 0 < x < 1, \tag{5}$$
$$u(0) = 0 \quad u'(1) = 0 \tag{6}$$

and prove that,

$$\int_0^1 ((u'(x))^2 + (u(x))^2)dx + \frac{1}{2}(u(1))^2 = \int_0^1 f(x)u(x)dx. \tag{7}$$

(b) Let $\Omega \subset \mathbb{R}^2$ open, bounded, $\vec{n}$ unit external normal, $\vec{a} \in \mathbb{R}^2$, $\vec{f} \in C^0(\overline{\Omega})$, $\Gamma_- = \{\vec{x} \in \partial\Omega; \vec{u} \cdot \vec{n} < 0\}$, assume there exists $u \in C^2(\overline{\Omega})$ such that

$$-\Delta\vec{u}(\vec{x}) + \vec{a} \cdot \nabla\vec{u}(\vec{x}) + \vec{u}(\vec{x}) = \vec{f}(\vec{x}) \quad \forall\vec{x} \in \Omega, \tag{8}$$
$$\vec{u}(\vec{x}) = 0 \quad \forall\vec{x} \in \Gamma_- \tag{9}$$
$$\frac{\partial\vec{u}(\vec{x})}{\partial\vec{n}} = 0 \forall\vec{x} \in \partial\Omega/\Gamma_- \tag{10}$$

and prove that,

$$\int_\Omega (||\nabla\vec{u}(x)||^2 + (\vec{u}(x))^2)dx + \int_{\Omega/\Gamma_-} \vec{a} \cdot \vec{n}\frac{\vec{u}(x)^2}{2}ds = \int_\Omega \vec{f}(\vec{x})\vec{u}(\vec{x})dx. \tag{11}$$

(c) Let $T > 0$, $\Omega \subset \mathbb{R}^2$ open, bounded, $\vec{n}$ unit external normal, $\vec{a} \in \mathbb{R}^2$, $\vec{u}_0 \in C^0(\overline{\Omega})$, $\vec{f} \in C^0(\overline{\Omega})$, $\Gamma_- = \{\vec{x} \in \partial\Omega; \vec{u} \cdot \vec{n} < 0\}$, assume there exists $\vec{u} \in C^{2,1}(\overline{\Omega} \times [0,T])$ (C^2 in space and C^1 in time) such that

$$\frac{\partial\vec{u}}{\partial t}(\vec{x},t) - \Delta\vec{u}(\vec{x},t) + \vec{a} \cdot \nabla\vec{u}(\vec{x},t) = 0 \quad \forall(\vec{x},t) \in \Omega \times [0,T], \tag{12}$$
$$\vec{u}(\vec{x},t) = 0 \quad \forall\vec{x} \in \Gamma_- \ 0 \leqslant t \leqslant T \tag{13}$$
$$\frac{\partial\vec{u}(\vec{x},t)}{\partial\vec{n}} = 0\forall\vec{x} \in \partial\Omega/\Gamma_- \ 0 \leqslant t \leqslant T \tag{14}$$
$$\vec{u}(\vec{x},0) = \vec{u}_0(\vec{x}) \quad \forall\vec{x} \in \Omega \tag{15}$$

and prove that,

$$\frac{1}{2}\int_\Omega (\vec{u}(\vec{x},T))^2 dx + \int_0^T \int_\Omega ||\nabla\vec{u}(\vec{x},t)||^2 dxdt + \int_0^T \int_{\Omega/\Gamma_-} \vec{a} \cdot \vec{n}\frac{\vec{u}(x)^2}{2}dsdt = \frac{1}{2}\int_\Omega (\vec{u}_0(\vec{x}))^2 dx. \tag{16}$$