EPFL - Advanced Numerical Analysis - Marco Picasso - Fall 2025

Problem Sheet 3

September 22, 2025

Question 1

Consider an s-stages implicit Runge-Kutta method for solving $y'(t) = f(t, y(t))$, $y(t_0) = y_0 \in \mathbb{R}^d$

$$k_i = f(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j) \quad i = 1, ..., s, \tag{1}$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i k_i$$

Here $c_i, b_i, a_{ij} \in \mathbb{R}$ and satisfy $c_1 = 0$, $c_i = \sum_{j=1}^{s} a_{ij}$.

Assume $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous (with constant L and norm $||\cdot||$) with respect to the second variable. Prove that there exists a unique solution of (1) if $h < \frac{1}{L \max\limits_{1 \leqslant i \leqslant s} \sum_{j=1}^{s} |a_{ij}|}$.

Indication : consider the fixed point iteration

$$k_i^{(m+1)} = f(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j^{(m)}) \quad m = 0, 1, 2, ... \tag{2}$$

and use Banach fixed point theorem.

Question 2

Graded exercise for group 3

Finish the proof of Theorem 1 to obtain the third order conditions.

Question 3

Consider the ordinary differential equation given by

$$\begin{cases} \dot{y}(t) = \lambda y(t), & 0 < t \leqslant T, \\ y(0) = y_0, \end{cases} \tag{3}$$

with $\lambda < 0$. Let N be a positive integer, let $h = \frac{T}{N}$ be the time step and $t_n = nh$ where $n = 0, 1, ..., N$. Consider now an order 4 RK scheme with 4 stages to approximate (3). Let

$$p_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

- Prove that $y_N = (p_4(\lambda h))^N y_0$.
- Prove that there exists $C > 0$ such that $\forall \lambda < 0, \forall h > 0$ such that $|p_4(\lambda h)| \leqslant 1$, $\forall T > 0$ and $\forall y_0 \in \mathbb{R}$,

$$|y(t_N) - y_N| \leqslant C |\lambda|^5 T h^4 |y_0|.$$