Problem Sheet 13

December 8, 2025

Question 1

Graded Exercise for Group 2 and Group 3

Consider the implicit schema for solving the wave equation in $[0,1]\times[0,T]$

$$\frac{\vec{u}^{n-1}-2\vec{u}^{n}+\vec{u}^{n+1}}{\tau^2}+A\frac{\vec{u}^{n-1}+2\vec{u}^{n}+\vec{u}^{n+1}}{4}=\vec{f}^n$$

where u_i^n is an approximation of $u(x_i,t_n)$, the exact solution of the wave equation, $x_i=ih$, $i=1,...,N$, $h=\frac{1}{N+1}$, $t_n=n\tau$, $n=1,...,M$ and $\tau=\frac{T}{M}$.

(a) Prove that $y^M\leqslant y^1+\tau\sum_{n=0}^{M-1}C^n$ where

$$y^M=\left(\left\|\frac{\vec{u}^M-\vec{u}^{M-1}}{\tau}\right\|^2+\left(\frac{\vec{u}^M+\vec{u}^{M-1}}{2}\right)^T A\left(\frac{\vec{u}^M+\vec{u}^{M-1}}{2}\right)\right)^{\frac{1}{2}}$$

and $C^n=||\vec{f}^n||$.

(b) Let $\vec{U}^n$ be the vector of components $u(x_i,t_n)$, prove that, if $\vec{u}\in C^4([0,1]\times[0,T])$

$$\frac{\vec{U}^{n-1}-2\vec{U}^{n}+\vec{U}^{n+1}}{\tau^2}+A\frac{\vec{U}^{n-1}+2\vec{U}^{n}+\vec{U}^{n+1}}{4}=\vec{f}^n+\vec{r}^n$$

with $|r_i^n|\leqslant C(h^2+\tau^2)$ and C independent of h and τ.

(c) Let $\vec{E}^n=\vec{U}^n-\vec{u}^n$ prove that : $z^M\leqslant z^0+\frac{CT}{\sqrt{h}}(h^2+\tau^2)$, where

$$z^M=\left(\left\|\frac{\vec{E}^M-\vec{E}^{M-1}}{\tau}\right\|^2+\left(\frac{\vec{E}^M+\vec{E}^{M-1}}{2}\right)^T A\left(\frac{\vec{E}^M+\vec{E}^{M-1}}{2}\right)\right)^{\frac{1}{2}}$$

Question 2

Consider the wave equation with $f=0$, $v_0=0$, $c=1$ and $T=1$.

(a) Implement the explicit scheme ($h=\dfrac{1}{N+1}$, $\tau=\dfrac{T}{M}$).

(b) When $u_0(x)=\sin(\pi x)$, check that $\max\limits_{1\leqslant i\leqslant N}|u_i^M-u(x_i,t_M)|=\mathcal{O}(h^2)$ when $\tau=h$.

(c) When $u_0(x)=e^{-1000(x-0.5)^2}$, run the scheme with $N=99,\ M=100$, then $N=199,\ M=200$. What happens when $N=199,\ M=199$?