Problem Sheet 12

December 1, 2025

Question 1

Graded Exercise for Group 3

Let $u :]0,1[^2 \to \mathbb{R}$ be such that :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,y,t) + \dfrac{\partial u}{\partial x}(x,y,t) = 0, \qquad (x,y) \in]0,1[^2, \qquad t > 0; \\ u(x,y,0) = 1, \qquad (x,y) \in]0,1[^2; \\ u(0,y,t) = 0, \qquad y \in]0,1[\qquad t > 0. \end{cases} \tag{1}$$

(a) What is the exact solution at $t = 0.5$?

(b) Fill the **trasp2d.m** file, run the code with $N = 99, 199, 399$ and $M = 60, 120, 240$ to check (eye) convergence.

(c) What happens whe $N = 99$ and $M = 50$?

Question 2

Given $u_0, v_0 : (0,1)^2 \to \mathbb{R}$, let u be the solution of the 2D wave equation with damping :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,y,t) - \Delta u(x,y,t) + \dfrac{\partial u}{\partial t}(x,y,t) = 0, \qquad (x,y) \in (0,1)^2, \qquad t > 0; \\ u(x,y,0) = u_0(x,y), \qquad (x,y) \in (0,1)^2; \\ \dfrac{\partial u}{\partial t}(x,y,0) = v_0(x,y), \qquad (x,y) \in (0,1)^2; \\ u(x,y,t) = 0, \qquad (x,y) \in \partial(0,1)^2 \qquad t > 0. \end{cases} \tag{2}$$

Prove that for all $t > 0$,

$$\frac{d}{dt}\left(\int_0^1 \int_0^1 \left(\left(\frac{\partial u}{\partial t}(x,y,t)\right)^2 + ||\nabla u(x,y,t)||^2\right)\right) dxdy \leqslant 0$$

Hint : multiply the wave equation by $\frac{\partial u}{\partial t}$ and integrate by parts.