Goal:

- Explicit Runge Kutta (RK4) ode's
- Finite Differences pde's

Course:

- lessons (pictures of the boards)
- 3 groups, 1 group each week

1 graded ex, upload Monday 10
(next week only Tuesday 10)
3 validated graded ex → 20% grade
Final exam Jan → 80% —
50 students → 4 failed
Motivation: why pde's

Part 1: E

Find $\vec{y}(t)$

where $\vec{y}_0$

$\begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix}$

Examples

- $\begin{cases} y'(t) \\ y(0) \end{cases}$

Part 1: Explicit RK methods (order 4) for ode's

Find $\vec{y}(t) \in \mathbb{R}^d$ such that

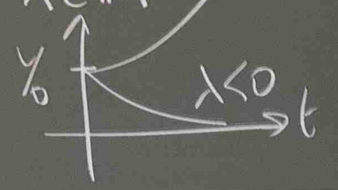
$$\begin{cases} \vec{y}'(t) = \vec{f}(t, \vec{y}(t)) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

where $\vec{y}_0 \in \mathbb{R}^d$ and $\vec{f}: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $(t, \vec{z}) \to \vec{f}(t, \vec{z})$ are known

$$\begin{pmatrix} y_1'(t) \\ \vdots \\ y_d'(t) \end{pmatrix} = \begin{pmatrix} f_1(t, y_1(t), \dots, y_d(t)) \\ \vdots \\ f_d(t, y_1(t), \dots, y_d(t)) \end{pmatrix}$$

Examples $d = 1$

- $\begin{cases} y'(t) = \lambda y(t) \\ y(0) = y_0 \end{cases}$ $\quad y(t) = y_0 e^{\lambda t}$



Ex: Newton's eq

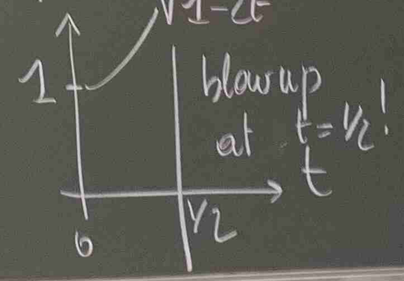
$$m \vec{X}''(t) = \vec{F}(t, \vec{X}(t), \vec{X}'(t))$$

$$\vec{X}'(t) = \vec{V}(t)$$

$$\vec{V}'(t) = \frac{1}{m} \vec{F}(t, \vec{X}(t), \vec{V}(t))$$

$$\vec{y}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ V_1(t) \\ V_2(t) \\ V_3(t) \end{pmatrix}$$

- $\begin{cases} y'(t) = (y(t))^3 \\ y(0) = 1 \end{cases}$ $\quad y(t) = \frac{1}{\sqrt{1-2t}}$



$y y'' = 1$

$m = -3$

$\frac{1}{m+1} y^{m+1} = t + C$

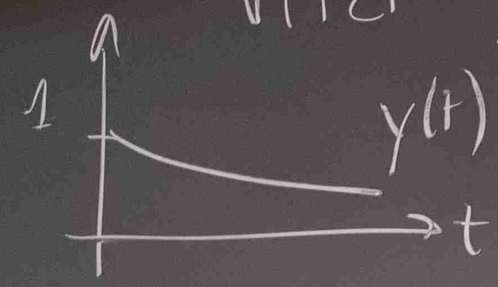
$$\begin{cases} y'(t) = -(y(t))^3 \\ y(0) = 1 \end{cases}$$

$y'y^n = -1$

$n = -3$

$$y(t) = \frac{1}{\sqrt{1+2t}}$$

one unique solution $\forall t$

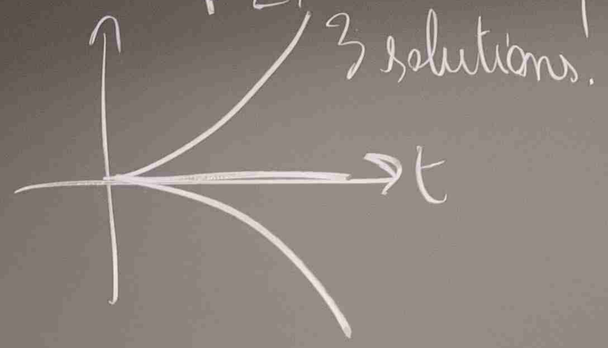


$$\begin{cases} y'(t) = (y(t))^{1/3} \\ y(0) = 0 \end{cases}$$

$y'y^n = 1$

$n = -1/3$

$$y(t) = \pm\sqrt{\frac{8t^3}{27}} \text{ and } 0$$



3 solutions!

$$\begin{cases} y'(t) = 3y(t) - 3t \\ y(0) = \alpha \end{cases}$$

$$y(t) = \left(\alpha - \tfrac{1}{3}\right)e^{3t} + t + \tfrac{1}{3}$$

unique sol but

$\alpha = 1/3$ $\quad y(10) = 10 + 1/3$

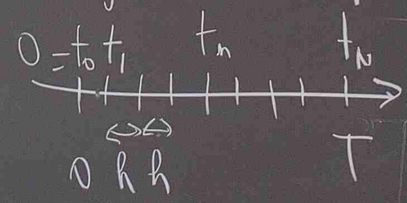
$\alpha = 1/3 + 10^{-6}$ $\quad y(10) = 10 + 10^{-6}e^{30} + 1/3$

$\simeq 10^7$

Thm:
- local existence: $f \in \mathcal{C}^1$ in the neighb. of (t_0, y_0) then $\exists \alpha > 0$ such that there exists a unique sol to $y'(t) = f(t, y(t))$, $y(t_0) = y_0$, for $t_0 - \alpha < t < t_0 + \alpha$
- Global existence: if f is lipschitz cont. with respect to the second variable $\exists L > 0 \; \forall t, z_1, z_2, \|f(t, z_1) - f(t, z_2)\| \leq L \|z_1 - z_2\|$

Then there exists a unique sol to
$y'(t) = f(t, y(t))$, $y(t_0) = y_0$

Euler scheme (explicit, implicit)
$d = 1$ $t_0 = 0$ final time T, integer N, time step $h = \frac{T}{N}$
$t_n = nh$ $n = 0, 1, \dots, N$



$E_n = y(t_n)$
$\frac{E_{n+1} - E_n}{h}$
If f is
$|E_{n+1}|$
Y_n approxim
Explicit E
Error E
Explicit

$E_n = y(t_n) - y_n$

$\frac{E_{n+1} - E_n}{h} = f(t_n, y(t_n)) - f(t_n, y_n) + \frac{h}{2} y''(s_n)$

$E_{n+1} = E_n + h(f \quad - f \quad) + \frac{h^2}{2} y''(s_n)$

If f is lipschitz cont. with resp to the sec. variable

$|E_{n+1}| \leq |E_n|(1+hL) + Ch^2 \qquad C = \frac{1}{2} \max_{0 \leq s \leq T} |y''(s)|$

$|E_{n+1}| \leq |E_{n-1}|(1+hL)^2 + Ch^2(1+(1+hL))$

$\vdots$

$\leq |E_0|(1+hL)^{n+1} + Ch^2(1+(1+hL)+\cdots+(1+hL)^n)$

y_n approximation of $y(t_n)$ $n = 1, \dots, N$

Explicit Euler: $\frac{y_{n+1} - y_n}{h} = f(t_n, y_n) \quad n = 0, \dots, N-1$

$y_{n+1} = y_n + h f(t_n, y_n)$

Error $E_N = y(t_N) - y_N = O(h)$ (order 1)

Explicit RK4 : $O(h^4)$ should be more precise

$y'(t_n) = f(t_n, y(t_n))$

$y(t_{n+1}) = y(t_n) + h y'(t_n) + \frac{h^2}{2} y''(s_n)$

If $y \in C^2[0,T]$ $\exists\, t_n < s_n < t_{n+1}$ such that

$y'(t_n) = \frac{y(t_{n+1}) - y(t_n)}{h} - \frac{h}{2} y''(s_n)$

$\frac{y(t_{n+1}) - y(t_n)}{h} = f(t_n, y(t_n)) + \frac{h}{2} y''(s_n)$

$$|E_N| \le Ch^2\left(1+(1+hL)+\dots+(1+hL)^{N-1}\right)$$

$$a^N - b^N = (a-b)(a^{N-1} + a^{N-2}b + \dots + ab^{N-2} + b^{N-1})$$

$b=1 \quad a = 1+hL$

$$|E_N| \le Ch^2 \frac{(1+hL)^N - 1}{1+hL-1} = \frac{Ch}{L}\left((1+hL)^N - 1\right)$$

$x>0 \quad e^x = 1+x+\frac{x^2}{2}e^{\xi} \quad 0<\xi<x$

$e^x \ge 1+x \qquad (1+x)^N \le e^{Nx}$

$$|E_N| \le \frac{C}{L}\left(e^{NhL} - 1\right)h = \frac{C}{L}\left(e^{LT} - 1\right)h$$

Lemma: If $y \in C^2[0,T]$, f lipschitz w.r. sec. var.
Then $\exists C>0 \quad \forall \; h>0$

$$|y(t_N) - y_N| \le Ch \quad \text{(we say } O(h) \text{ order 1)}$$

$$\left(C = \max_{0\le\xi\le T} |y''(\xi)| \frac{\left(e^{LT}-1\right)}{2L}\right)$$

$|E_{n-1}|(1+hL)^2$

$Ch^2(1+(1+hL))$

$(1+hL)^{n-1}$

$+(1+hL)+\dots+(1+hL)^n)$

$, y(t_n))$

$y'(t_n) + \frac{h}{2}y''(\xi_n)$.

$t_n < \xi_n < t_{n+1}$

such that

$-y(t_n) - \frac{h}{2}y''(\xi_n)$

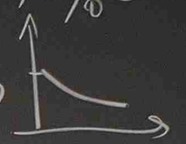
h

$= f(t_n, y(t_n)) + \frac{h}{2}y''(\xi_n)$

Issue: e^{LT} can be large

This convergence result is not optimal in the linear case:

Lemma: $\begin{cases} y'(t) = \lambda y(t) \\ y(0) = y_0 \end{cases}$ $\lambda < 0$ $\quad (y(t) = y_0 e^{\lambda t})$



Euler implicit: $\frac{y_{n+1} - y_n}{h} = \lambda y_{n+1}$

$\exists C > 0 \; \forall T > 0 \; \forall \lambda < 0 \; \forall h > 0 \quad |y(t_N) - y_N| \leq C T \lambda^2 h$

Euler explicit: $\frac{y_{n+1} - y_n}{h} = \lambda y_n$ stability condition

$\exists C > 0 \; \forall T > 0 \; \forall \lambda < 0 \; \forall 0 < h < \frac{2}{-\lambda} \quad |y(t_N) - y_N| \leq C T \lambda^2 h$

Proof: explicit; $y(t_N) = y_0 e^{\lambda N h}$

$y_{n+1} = y_n (1 + \lambda h) = y_0 (1 + \lambda h)^{n+1}$

$y_N = y_0 (1 + \lambda h)^N$

Finally

C =

Implicit

$y(t_N$

$x > 0$

Finally $|y(t_N) - y_N| \leq |y_0| \frac{\lambda^2 h^2}{2} N = |y_0| \frac{\lambda^2 T}{2} h$

$C = \frac{|y_0|}{2}$

Implicit case: $\frac{y_{n+1} - y_n}{h} = \lambda y_{n+1}$

$y_{n+1} = y_n \frac{1}{1-\lambda h}$

$y_{n+1} = y_0 \frac{1}{(1-\lambda h)^{n+1}}$

$y(t_N) - y_n = y_0 \left(e^{\lambda h N} \frac{1}{(1-\lambda h)^N} \right)$

$= y_0 \left(e^{\lambda h} - \frac{1}{1-\lambda h} \right) \left(e^{\lambda h (N-1)} + \dots + \frac{1}{(1-\lambda h)^{N-1}} \right)$

$x > 0 \quad \frac{1}{1+x} \leq 1$

$x > 0 \quad \left| e^{-x} - \frac{1}{1+x} \right| \leq x^2 \frac{3}{2}$

$y(t_N) - y_N = y_0 \left(e^{\lambda h N} - (1+\lambda h)^N \right)$

$= y_0 \left(e^{\lambda h} - (1+\lambda h) \right) \left(e^{\lambda h (N-1)} + \dots + (1+\lambda h)^{N-1} \right)$

$x > 0 \quad e^{-x} = 1 - x + \frac{x^2}{2} e^{-s} \quad 0 < s < x$

$\left| e^{-x} - (1-x) \right| = \left| \frac{x^2}{2} e^{-s} \right| = \frac{x^2}{2} e^{-s} \leq \frac{x^2}{2}$

Therefore $|y(t_N) - y_N| \leq |y_0| \frac{\lambda^2 h^2}{2} \left(\underbrace{\left| e^{\lambda h (N-1)} \right| + \dots + \left| 1 + \lambda h \right|^{N-1}}_{N \text{ terms}} \right)$

$\left| e^{\lambda h (N-1)} \right| = e^{\lambda h (N-1)} \leq 1 \quad (\lambda < 0)$

$\left| 1 + \lambda h \right|^{N-1} \leq 1$

if $|1 + \lambda h| \leq 1$

$-1 \leq 1 + \lambda h \leq 1$

$-\lambda h \leq 2$

$h \leq \frac{2}{-\lambda}$

$$\begin{cases} \vec{y}'(t) = \vec{f}(t, y(t)) \in \mathbb{R}^d \\ y(t_0) = \vec{y}_0 \end{cases}$$

Euler scheme $h > 0$ Y_1 approx $y(\underbrace{t_0 + h}_{t_1})$

$$Y_1 = Y_0 + h f(t_0, Y_0)$$

$y(t_0+h) - Y_1 = O(h^2)$ t_N final time

$y(t_N) - Y_N = O(h)$ order 1

higher order? Explicit Runge Kutta order 4

Runge scheme (order 2)

$$y'(t_0 + h/2) = f(t_0 + h/2, y(t_0 + h/2))$$

$$\frac{y(t_0+h) - y(t_0)}{h} = f\left(t_0 + h/2, y(t_0 + \tfrac{h}{2})\right) + O(h^2)$$

$$= f\left(t_0 + h/2, y(t_0) + \frac{h}{2} f(t_0, y(t_0))\right) + O(h^2)$$

$$y(t_0+h) = y(t_0) + h f\left(t_0 + h/2,\ y(t_0) + \frac{h}{2} f(t_0, y(t_0))\right) + O(h^3)$$

$$y_1 = y_0 + h f\left(t_0 + h/2,\ y_0 + \frac{h}{2} f(t_0, y_0)\right)$$

$$y(t_0+h) - y_1 = O(h^3)$$

$$y(t_N) - y_N = O(h^2)$$

Runge Kutta uniting

$$\vec{k}_1 = \vec{f}(t_0, \vec{y}_0)$$

$$\vec{k}_2 = \vec{f}\left(t_0 + \frac{h}{2},\ \vec{y}_0 + \frac{h}{2}\vec{k}_1\right)$$

$$\vec{y}_1 = \vec{y}_0 + h\vec{k}_2$$

(evaluating f is costly, $\phi = 10^6$)

s shages explicit Runge Kutta method

$$k_1 = f(t_0, y_0)$$

$$k_2 = f(t_0 + c_2 h,\ y_0 + h a_{21} k_1)$$

$$\vdots$$

$$k_s = f\left(t_0 + c_s h,\ y_0 + h\left(a_{s1} k_1 + \cdots + a_{s,s-1} k_{s-1}\right)\right)$$

$$y_1 = y_0 + h\left(b_1 k_1 + \cdots + b_s k_s\right)$$

table

$c_1 = 0$			
c_2	a_{21}		
$\vdots$	$\vdots$		
c_s	a_{s1}	$\cdots$	$a_{s,s-1}$
	b_1	$\cdots$	b_s

Euler order 1

0	
	1

Runge order 2

0		
$\frac{1}{2}$	$\frac{1}{2}$	
	0	1

ERK4 order 4

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
1	0	0	1	
	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

Rem: • there are not Runge Kutta methods of order 5 with 5 stages!

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ c_2 & & & \\ \vdots & & & \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

defines an implicit RK method

Ex: Crank-Nicolson

$$y_1 = y_0 + \frac{h}{2}\left(f(t_0, y_0) + f(t_1, y_1)\right)$$

$k_1 = f(t_0, y_0)$

~~$k_2 = f(t_0 + h, y_1)$~~

$k_2 = f\left(t_0 + h, y_0 + h\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)\right)$ $k_2 \to$ solve a nonlin eq.

$y_1 = y_0 + h\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)$

Def: the s stages RK method (impl. or expl.) is of order p if the error after on step is

$$\underbrace{y(t_0 + h) - y_1}_{\text{local error}} = O(h^{p+1})$$

Question: • how to choose the c_i, a_{ij}, b_i so that the error is of order $p = 4$ (Thm 1)

• if the local error (error after on step) is $O(h^{p+1})$ is the final error $O(h^p)$? (Thm 2)

Thm 1: $y' = f(y)$ $(y'(t) = f(y(t))$ autonomous case
reason why ex)

$1 \le p \le 4$ $\quad y(t_0 + h) - y_1 = O(h^{p+1})$ if and only if

$\sum_{i=1}^{s} b_i = 1$ $\quad p=1$

$\sum_{i,j=1}^{s} b_i a_{ij} = 1/2$ $\quad p=2$

$\sum_{i,j,k=1}^{s} b_i a_{ij} a_{ik} = 1/3$ $\quad \sum_{i,j,k=1}^{s} b_i a_{ij} a_{jk} = 1/6$ $\quad p=3$

$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{ik} a_{il} = 1/4$

$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} a_{il} = 1/8$

$\sum_{i,j,k,l}^{s} b_i a_{ij} a_{jk} a_{jl} = 1/12$

$\sum_{i,j,k,l}^{s} b_i a_{ij} a_{jk} a_{kl} = 1/24$

$p=4$

These conditions are valid for impl. and expl RK methods

When $c_1 = 0$ and $c_i = \sum_{j=1}^{s} a_{ij}$ $i = 1, \dots, s$ (the $y' = f(y)$ assumption can be justified)

these conditions simplify to

$$\sum_i b_i = 1 \qquad p=1$$

$$\sum_i b_i c_i = 1/2 \qquad p=2$$

$$\sum_i b_i c_i^2 = 1/3 \qquad \sum_{ij} b_i a_{ij} c_j = 1/6 \qquad p=3$$

$$\sum_i b_i (c_i)^3 = 1/4$$

$$\sum_{ij} b_i a_{ij} c_j c_i = 1/8$$

$$\sum_{ij} b_i a_{ij} c_j^2 = 1/12$$

$$\sum_{ijk} b_i a_{ij} a_{jk} c_k = 1/24$$

It can be checked that for $s = 4$

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	2/6	2/6	1/6

satisfies these conditions

no scheme order 5 with 5 stages

Proof thm 1:

order 1: $y(t_0+h) = y(t_0) + h y'(t_0) + O(h^2)$ (1)

$= y_0 + h f(y_0) + O(h^2)$

$$y_1(h) = y_0 + h \underbrace{\sum_{i=1}^{s} b_i k_i(h)}_{\phi(h)}$$

$$k_i(h) = f\Big(y_0 + \underbrace{h \sum_{j=1}^{s} a_{ij} k_j(h)}_{\psi_i(h)}\Big) \qquad i = 1, \dots, s$$

$$y_1(h) = y_0 + \phi(0) + h\phi'(0) + O(h^2) \quad (2)$$

$$\phi'(h) = \sum_{i=1}^{s} b_i k_i(h) + h \sum_{i=1}^{s} b_i k_i'(h)$$

$$\phi'(0) = \sum_{i=1}^{s} b_i f(y_0)$$

$$(1)-(2) \cdot \vec{y}(t_0+h) - \vec{y}_1(h) = h\underbrace{\left(1 - \sum_{i=1}^{s} b_i\right)}_{=0}\vec{f}(y_0) + O(h^2)$$

order 2

$$y(t_0+h) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + O(h^3)$$

$$y_i'(t) = f(y_1(t), \dots, y_d(t)) \quad i = 1, \dots, d$$

$$y_i''(t) = \sum_{j=1}^{d} \frac{\partial f}{\partial y_j}(\vec{y}(t))\, y_j'(t)$$

$$\vec{y}'' = f'(y)\vec{y}' \qquad f'(y): \mathbb{R}^d \to \mathbb{R}^d, \quad \vec{u} \to f'(y)\vec{u}$$

$$\left(f'(y)\vec{u}\right)_i = \sum_{j=1}^{d} \frac{\partial f}{\partial y_j}(y) u_j$$

$$y''(t_0) = f'(y_0) f(y_0)$$

$\in \mathbb{R}^d$ — matrix — $\in \mathbb{R}^d$

$$y(t_0+h) = y_0 + h f(y_0) + \frac{h^2}{2} f'(y_0) f(y_0) + O(h^3)$$

New term in (2): $\frac{h^2}{2} \phi''(0) + O(h^3)$

$$\phi''(h) = 2 \sum_{i=1}^{s} b_i k_i'(h) + h \sum_{i=1}^{s} b_i k_i''(h)$$

$$k_i'(h) = f'(y_0 + \Psi_i(h)) \Psi_i'(h)$$

$$\Psi_i'(h) = \sum_j a_{ij} k_j(h) + h \sum_j a_{ij} k_j'(h)$$

$$\phi''(0) = 2 \sum_{i=1}^{s} b_i f'(y_0) \sum_{j=1}^{s} a_{ij} f(y_0)$$

$$= 2 \sum_{i,j=1}^{s} b_i a_{ij} f'(y_0) f(y_0)$$

$$y''(t_0) - \phi''(0) = \underbrace{\left(1 - 2 \sum_{ij} b_i a_{ij}\right)}_{0} f'(y_0) f(y_0)$$

order 3: extra term in (1):

$$\frac{h^3}{6} y'''(t_0) + O(h^4)$$

$$y_i'''(t) = \sum_{j,k=1}^{d} \frac{\partial^2 f}{\partial y_j \partial y_k}(y(t))\, y_j'(t) y_k'(t) + \sum_{j=1}^{d} \frac{\partial f}{\partial y_j}(y(t))\, y_j''(t) \qquad i = 1, \dots, d$$

$$y''' = f''(y)(y', y') + f'(y) y''$$

$f'(y) f'(y) y'$ — matrix matrix vector

$f''(y) f'(y) y'$ — matrix vector

vector

$$f''(y): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \quad (u, v) \to f''(y)(u, v)$$

$$\big(f''(y)(u,v)\big)_i := \sum_{j,k} \frac{\partial^2 f}{\partial y_j \partial y_k}(y(t))\, u_j v_k$$

$$y'''(t_0) = f''(y_0)\big(f(y_0), f(y_0)\big) + f'(y_0) f'(y_0) f(y_0)$$

extra term in (2):

$$\frac{h^3}{6} \phi'''(0) + O(h^4)$$

$$\phi'''(h) = 3 \sum_i b_i k_i''(h) + h \sum_i b_i k_i'''(h)$$

$$k_i''(h) = f''(y_0 + \Psi_i(h))\big(\Psi_i'(h), \Psi_i'(h)\big) + f'(y_0 + \Psi_i(h))\, \Psi_i''(h)$$

$$\Psi_i''(h) = 2 \sum_j a_{ij} k_j'(h) + h \sum_j a_{ij} k_j''(h)$$

$$\Psi_i''(0) = 2 \sum_j a_{ij} f''(y_0) f'(y_0) \sum_k a_{jk} f(y_0)$$

$$k_i''(0) = f''(y_0)\Big(\sum_j a_{ij} f(y_0), \sum_k a_{ik} f(y_0)\Big) + f''(y_0) 2 \sum_j a_{ij} f'(y_0) \sum_k a_{jk} f(y_0)$$

$$\phi'''(0) = 3 \sum_i b_i$$

$$y'(t) = f(t, y(t)) \quad \bar{y}(t) \in \mathbb{R}^d$$
$$y(t_0) = y_0$$

RK: y_1 approx $y(t_0+h)$ $h>0$

$$k_i = f\left(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j\right) \quad i = 1, \dots, s$$

(s) i-1 explicit

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i k_i$$

- order conditions $y(t_0+h) - y_1 = O(h^{p+1})$ $p=4$
- adaptive time stepping h?
- convergence $y(t_n) - y_n = O(h^p)$?

---

adaptive time stepping

Let $h>0$ compute y_1 approx of $y(t_0+h)$ order p: $y(t_0+h)-y_1 = O(h^{p+1})$

$\hat{y}_1$ ——— $p-1$ ——— $\hat{y}_1 = O(h^p)$

We have: $|y_1-\hat{y}_1| - |y(t_0+h)-y_1| \le |y(t_0+h)-\hat{y}_1| \le |y(t_0+h)-y_1| + |y_1-\hat{y}_1|$

computable $O(h^{p+1})$ $O(h^p)$ $O(h^{p+1})$ computable

Therefore, if h is small enough, then $|y_1-\hat{y}_1|$ is a good approx of $y(t_0+h)-\hat{y}_1$

For a given h, we know that $|(y_1-\hat{y}_1)(h)| = Ch^p$, thus $C = \dfrac{|(y_1-\hat{y}_1)(h)|}{h^p}$

We are looking for h_{opt} thus that $(y_1-\hat{y}_1)(h_{opt}) = TOL$, where TOL is a given tolerance

But $(y_1-\hat{y}_1)(h_{opt}) = Ch_{opt}^p$ but we know C

$Ch_{opt}^p = TOL$

$(y_1-\hat{y}_1)(h)\left(\dfrac{h_{opt}}{h}\right)^p = TOL$ that is $h_{opt} = h\left(\dfrac{TOL}{(y_1-\hat{y}_1)(h)}\right)^{1/p}$

Algorithme (heuristic)

For each time step (h is current time step)

Compute y_1, $\hat{y}_1$, $|y_1 - \hat{y}_1|$, $h_{opt} = h\left(\frac{TOL}{|y_1 - \hat{y}_1|}\right)^{1/p}$

set $h = h_{opt}$

If $|y_1 - \hat{y}_1| \leq TOL$ goto the next time step

Else repeat the time step

Given y_1, the computation of $\hat{y}_1$ should be cheap

$p = 2$ y_1 Runge scheme

0	
1/2	1/2
	0 | 1

$\hat{y}_1$ Euler

$k_1 = f(t_0, y_0)$

$k_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} k_1)$

$y_1 = y_0 + h k_2$

$\hat{y}_1 = y_0 + h k_1$

$y_1 - \hat{y}_1 = h(k_2 - k_1)$

$p=4$: y_1 is computed with an order 4 ERK

c_1				
c_2	a_{21}			
c_3	a_{31}	a_{32}		
1	a_{41}	a_{42}	a_{43}	
	b_1	b_2	b_3	b_4

c_i, a_{ij}, b_i satisfy the order 4 cond. of thm 1

$$y_1 = y_0 + h(b_1 k_1 + \dots + b_4 k_4)$$

y_1 is such that $y(t_0+h) - y_1 = O(h^4)$ (order 3) keeping the same coefficients whenever possible, in order to avoid computing $f(\cdot,\cdot)$. Choose $s=5$

c_1					
c_2	a_{21}				
c_3	a_{31}	a_{32}			
1	a_{41}	a_{42}	a_{43}		
1	b_1	b_2	b_3	b_4	
	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$

needs only one more eval. of f

$$k_5 = f(t_0 + h, y_n + h(b_1 k_1 + \dots + b_4 k_4))$$

$$\hat{y}_1 = y_0 + h(\hat{b}_1 k_1 + \dots + \hat{b}_5 k_5)$$

The coeff $\hat{b}_1, \dots, \hat{b}_5$ are such that the order 3 cond. are satisfied (thm 1)

$$\begin{cases} \hat{b}_1 + \hat{b}_2 + \hat{b}_3 + \hat{b}_4 + \hat{b}_5 = 1 \\ \hat{b}_2 c_2 + \hat{b}_3 c_3 + \hat{b}_4 + \hat{b}_5 = 1/2 \\ \hat{b}_2 c_2^2 + \hat{b}_3 c_3^2 + \hat{b}_4 + \hat{b}_5 = 1/3 \\ \hat{b}_2 a_{32} c_2 + \hat{b}_4 (a_{42} c_2 + a_{43} c_3) + \hat{b}_5 \frac{1}{2} = \frac{1}{6} \end{cases}$$

Set $\hat{b}_5$, and solve the linear system to find $\hat{b}_1, \dots, \hat{b}_4$

(to be done once for all the time steps)

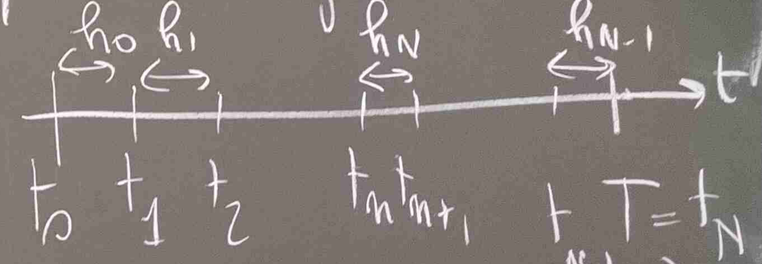
Convergence of the method

Write an s stages explicit Runge Kutta method as

$$y_{n+1} = y_n + h_n \phi(t_n, y_n, h_n)$$

$$t_{n+1} = t_n + h_n$$

ϕ will be defined later



$$h_{max} = \max(h_0, h_1, \dots, h_{N-1})$$

Thm 2: (convergence of ERK)

Assume that:

– the local error is of order p:

$$\exists C>0 \ \forall t_0 \le t \le T \ \forall 0<h \le h_{max}$$

$$\| y(t+h) - \big(y(t) + h\,\phi(t, y(t), h)\big) \| \le C\, h^{p+1}$$

– the function ϕ is lipschitz continuous with respect to the second variable

$$\exists \Lambda>0 \ \forall h \le h_{max} \ \forall (t, z_1) \text{ and } (t, z_2) \text{ in the neighb. of } (t, y(t))$$

$$\| \phi(t, z_1, h) - \phi(t, z_2, h) \| \le \Lambda \| z_1 - z_2 \|$$

Then the global error satisfies $n = 1, \dots, N$

$$|y(t_n) - y_n| \le \frac{C}{\Lambda}\left(e^{\Lambda(t_n - t_0)} - 1\right) h_{max}^{p}$$

Proof: $y(t_{n+1}) - y_{n+1} = y(t_{n+1}) - \left(y_n + h_n \phi(t_n, y_n, h_n)\right)$

$= y(t_{n+1}) - \left(y(t_n) + h_n \phi(t_n, y(t_n), h_n\right)$

$+ \left(\quad \right)$

$- \left(y_n + h_n \phi(t_n, y_n, h_n)\right)$

thus $\| y(t_{n+1}) - y_{n+1} \|$

$\leq C h_n^{p+1} + \| y(t_n) - y_n \| \left(1 + h_n \wedge \right)$

$\leq C h_n^{p+1} + \| y(t_n) - y_n \| e^{h_n \wedge}$

$\leq C h_n^{p+1} + C h_{n-1}^{p+1} e^{h_n \wedge} + \| y(t_{n-1}) - y_{n-1} \| e^{(h_n + h_{n-1}) \wedge}$

$\cdots\cdots$

$\leq C h_n^{p+1} + C h_{n-1}^{p+1} e^{h_n \wedge} + \cdots\cdots + C h_0^{p+1} e^{(h_n + \cdots h_1) \wedge} + \underbrace{\| y(t_0) - y_0 \|}_{0} e^{(h_n + \cdots + h_1 + h_0) \wedge}$

$x > 0$

$e^x = 1 + x + \frac{x^2}{2} e^{\theta x}$

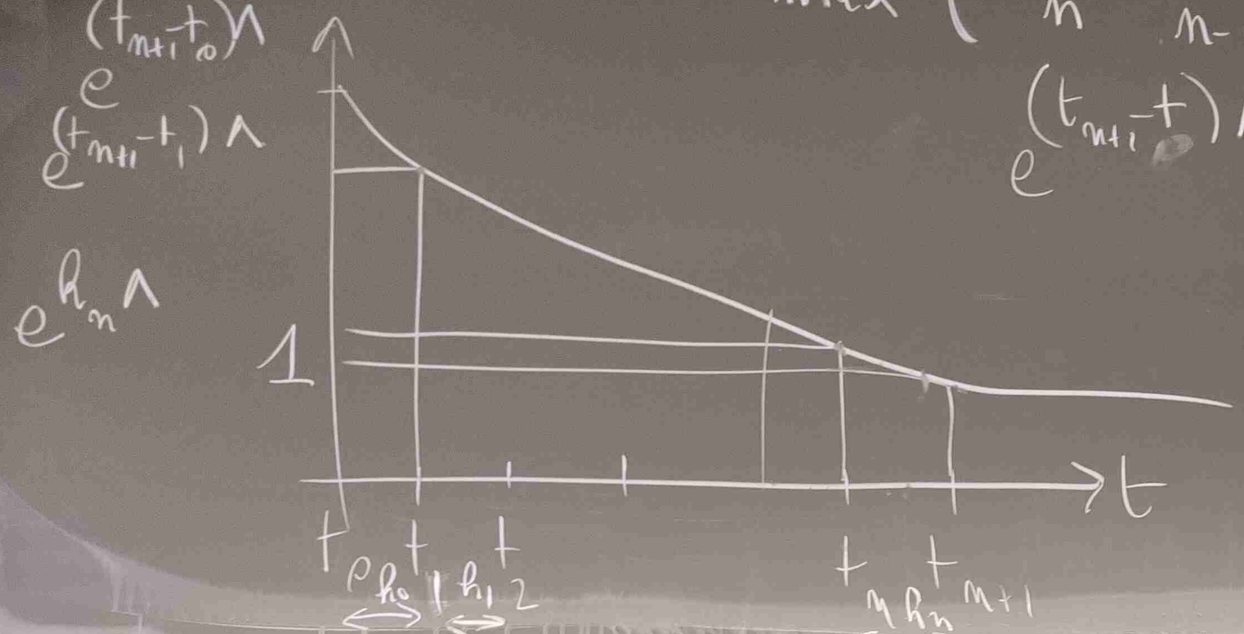
$\geq 1 + x \quad \big|_{0 < \theta x < 1}$

$$\|y(t_{n+1}) - y_{n+1}\| \leq C h_{max}^p \left(h_n + h_{n-1} e^{h_n \wedge} + \dots + h_0 e^{(h_n + \dots + h_1)\wedge} \right)$$

$e^{(t_{n+1}-t_0)\wedge}$

$e^{(t_{n+1}-t_1)\wedge}$

$e^{h_n \wedge}$



$e^{(t_{n+1}-t)\wedge}$

$$e^{(t_{n+1}-t_0)\wedge} \quad (t_{n+1}-t_n+t_n-t_{n-1}+\dots+t_1-t_0)\wedge$$
$$= e^{(h_n+h_{n-1}+\dots+h_0)\wedge}$$
$$= e^{\int_{t_0}^{t_{n+1}} (t_{n+1}-t)\wedge \, dt}$$

Finally $\|y(t_{n+1}) - y_{n+1}\| \leq C h_{max}^p \int_{t_0}^{t_{n+1}} e^{(t_{n+1}-t)\wedge} dt = \frac{C h_{max}^p}{\wedge} \left(e^{(t_{n+1}-t_0)\wedge} - 1 \right)$

does (t_n, y_n) remain in a neighb. of $(t_n, y(t_n))^{t_0}$?

yes, if $\frac{C h_{max}^p}{\wedge} \left(e^{(t_n - t_0)\wedge} - 1 \right)$ is small enough

which is the case if h_{max} is small enough

$y_{n+1} = y_n + h_n \phi(t_n, y_n, h_n)$ What is ϕ?

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i k_i \,, \quad k_i = f\Big(t_0 + c_i h, \underbrace{y_0 + h \sum_{j=1}^{i-1} a_{ij} k_j}_{i=1,\dots,s}\Big)$$

thus $\phi(t, z, h) = z + h \sum_{i=1}^{s} b_i k_i(t, z, h)$

$$k_i(t, z, h) = f\Big(t + c_i h, z + h \sum_{j=1}^{i-1} a_{ij} k_j(t, z, h)\Big) \quad i = 1, \dots, s$$

Lemma: if f lipsch. cont with resp to the sec. var

$\exists L > 0$ $\forall (t, z_1)$ and (t, z_2) in the neighb of $(t, y(t))$

$\| f(t, z_1) - f(t, z_2) \| \le L \| z_1 - z_2 \|$

then $\exists \Lambda$ (depon $h_{max}, L, c_i, a_{ij}, b_i$) such that

$\forall 0 \le t \le T$, $\forall 0 < h \le h_{max}$, $\forall (t, z_1)$ and (t, z_2) neighb of $(t, y(t))$

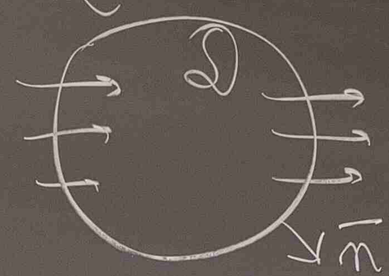
$\| \phi(t, z_1, h) - \phi(t, z_2, h) \| \le \Lambda \| z_1 - z_2 \|$

Part 2: Finite differences for pde's

1 - Introduction, motivation

2 - Elliptic problems 1d

1 - $\bar{v} : \mathbb{R}^3 \to \mathbb{R}^3$ C^1



Ω: arbitrary bounded domain in $\mathbb{R}^3$

$\int_{\partial\Omega} \bar{v} \cdot \bar{n} \, ds = 0$

divergence thm: $\int_{\Omega} \operatorname{div} \bar{v} \, d\bar{x} = 0$

Since Ω is arbitrary: $\operatorname{div} \bar{v} = 0$ in $\mathbb{R}^3$

Application:

II - electrostatics: $\bar{E}$ electrical field $\bar{E}$, density of charges ρ

$\int_{\partial\Omega} \bar{E} \cdot \bar{n} \, ds = \int_{\Omega} \rho \, dx$

$\operatorname{div} \vec{E} = \rho$

Since $\operatorname{curl} \vec{E} = \vec{0}$

$\vec{\nabla} \times \vec{E} = \vec{0}$ $\quad \vec{\nabla} = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix}$

$\vec{E} = -\vec{\nabla} \varphi$

thus $-\operatorname{div} \vec{\nabla} \varphi = \rho$

pbm to be solved:



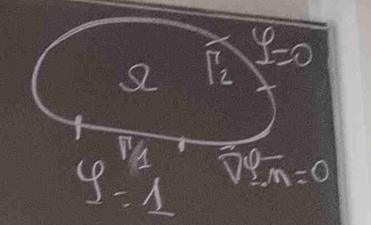
Ω domain $\subset \mathbb{R}^3$

$\rho : \Omega \to \mathbb{R}$ given

Find $\varphi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\underbrace{\operatorname{div} \vec{\nabla} \varphi}_{\Delta \varphi} = \rho & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

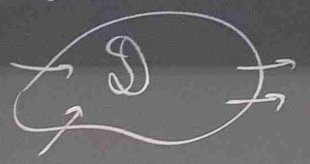
or



Find $\varphi : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} -\Delta \varphi = \rho & \text{in } \Omega \\ \varphi = 1 & \text{on } \Gamma_1 \\ \varphi = 0 & \text{on } \Gamma_2 \\ \vec{\nabla}\varphi \cdot \vec{n} = 0 & \text{on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2) \end{cases}$$

– heat transfer



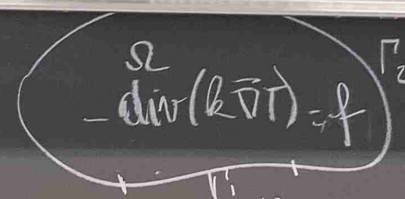
$\vec{q}$ heat flux $\quad f$: source term

$$\int_{\partial\Omega} \vec{q} \cdot \vec{n} = \int_{\Omega} f \, dx$$

$\operatorname{div} \vec{q} = f$

$\vec{q} = -k \vec{\nabla} T$ $\quad k > 0$ diffusion coeff

T: temperature



$k \vec{\nabla} T \cdot \vec{n} =$ ——— else

Find $T : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} -\operatorname{div}(k \vec{\nabla} T) = f & \text{in } \Omega \\ T = T_1 & \text{on } \Gamma_1 \\ k \vec{\nabla} T \cdot \vec{n} = q_2 & \text{on } \Gamma_2' \end{cases} \qquad \partial\Omega = \Gamma_1 \cup \Gamma_2'$$

Model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad d = 1, 2, 3$$

$d = 1$ given $f : [0,1] \to \mathbb{R}$

find $u : [0,1] \to \mathbb{R}$ s.t.

$$\begin{cases} -u''(x) = f(x) & 0 < x < 1 \\ u(0) = 0, \; u(1) = 0 \end{cases}$$

- heat equation (heat transfer with time)
$\Omega \subset \mathbb{R}^3$ arbitrary; conservation principle

$$\frac{d}{dt}\int_\Omega \underbrace{\rho c_p T}_{\text{enthalpy}}\,dx - \int_{\partial\Omega} k\nabla T\cdot\vec{n}\,ds = \int_\Omega f\,dx$$

$\int_\Omega \rho c_p T\,dx$: thermal energy in Ω — $\int_{\partial\Omega} k\nabla T\cdot\vec{n}\,ds$: heat flux across $\partial\Omega$ — $\int_\Omega f\,dx$: source or sink of thermal energy

$$\int_\Omega \left(\frac{\partial}{\partial t}(\rho c_p T) - \operatorname{div}(k\nabla T) - f\right)dx = 0 \quad \text{for all } \Omega$$

$$\frac{\partial}{\partial t}(\rho c_p T) - \operatorname{div}(k\nabla T) = f$$

$\rho c_p = 1 = k$, $d = 1$, $T = u(x,t)$: $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f$ parabolic eq.

- Wave eq (electromagnetics)

Model problem

$$\begin{cases} -\Delta u = f \text{ in } \Omega \subset \mathbb{R}^d, \ d=1,2,3 \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

$d = 1$, given $f:[0,1]\to\mathbb{R}$, find $u:[0,1]\to\mathbb{R}$ s.t.

$$\begin{cases} -u''(x) = f(x) & 0<x<1 \\ u(0)=0,\ u(1)=0 \end{cases}$$

Maxwell eq (vacuum)

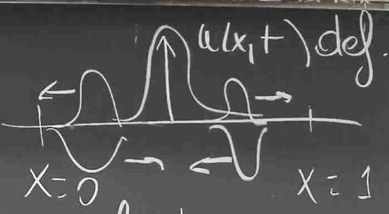
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} = -\frac{\partial}{\partial t} \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Since $\operatorname{div} \vec{E} = 0$ $\quad \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \Delta \vec{E} = \vec{0}$

$d = 1 \quad E = u \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f$

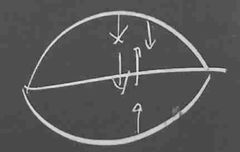


elastic rope

wave equation

multiply by $\frac{\partial u}{\partial t}$ and $\int_0^1 dx$

$$\frac{1}{2} \frac{d}{dt} \left(\underbrace{\int_0^1 \left(\frac{\partial u}{\partial t}\right)^2 dx}_{\text{kinetic energy}} + \underbrace{\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx}_{\text{def energy}} \right) = 0 \quad \text{if } f = 0$$



2_ Elliptic problems

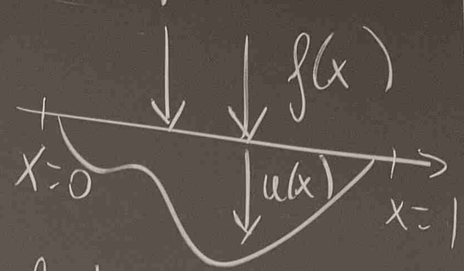
d=1

Given $f:[0,1]\to\mathbb{R}$

find $u:[0,1]\to\mathbb{R}$ s.t.

$$\begin{cases} -u''(x) = f(x) & 0<x<1 \\ u(0)=0 \quad u(1)=0 \end{cases}$$

Model problem



elastic rope

Remarks:

* exact solution:

$$u(x) = x\int_0^1 dt \int_0^t f(s)ds - \int_0^x dt \int_0^t f(s)ds$$

if $f\in C^0[0,1]$ then $u\in C^2[0,1]$

$$u(x) = (1-x)\int_0^x t f(t)\,dt + x\int_x^1 (1-t)f(t)\,dt$$

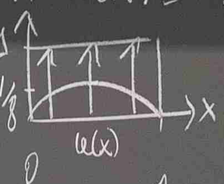
$$= \int_0^1 G(x,t) f(t)\,dt \quad G: \text{Green's fct}$$

$$G(x,t) = (1-x)t \quad 0\le t\le x$$

$$= x(1-t) \quad x\le t\le 1$$

If $f\ge 0$ then $u\ge 0$ and $\|u\|_{L^\infty(0,1)} \le \frac{1}{8}\|f\|_{L^\infty(0,1)}$

* $u(x) = \frac{x(1-x)}{2}$ if $f(x)=1$



* If $f(x) = \sum_{k=1}^{N} a_k \sin(k\pi x)$

then $u(x) = \sum_{k=1}^{N} \frac{a_k}{k^2\pi^2}\sin(k\pi x)$

linked to: if $u(x) = \sin(k\pi x)$

then $-u''(x) = k^2\pi^2 u(x)$

$\sin(k\pi x)$ eigenvector of $-u'' = \lambda u$

eigenvectors are $k^2\pi^2$

* if we multiply $-u'' = f$ by u and $\int_0^1 dx$, we have $\int_0^1 (u'(x))^2 dx = \int_0^1 f(x)u(x)dx$

deformation energy — work forces

prove uniqueness

(2 solution u_1 and u_2)

then $\int_0^1 (u_1'(x) - u_2'(x))^2 dx = 0$

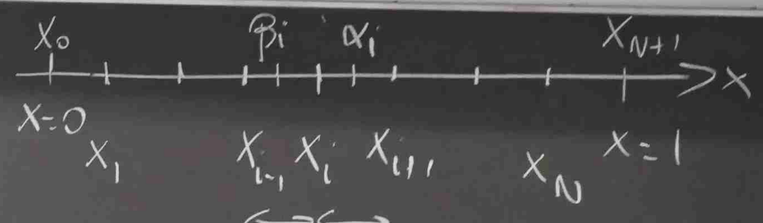
implies that $u_1 = u_2$

existence

in $\{v \in L^2(0,1);\ v' \in L^2(0,1)$

$v(0) = 0\ v(1) = 0\}$

Finite Differences:



$x_i = ih \quad i = 0, \dots, N+1$

Goal: to approximate $u(x_i)$ by a computable approx u_i, $i = 1, \dots, N$

Taylor expansion: $u \in C^4[0,1]$

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}u'''(x_i) + \frac{h^4}{24}u^{(4)}(\alpha_i)$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}u'''(x_i) + \frac{h^4}{24}u^{(4)}(\beta_i)$$

where $x_i < \alpha_i < x_{i+1}$

$(-u''(x_i) = f(x_i)\ i = 1, \dots, N)$

$$-u''(x_i) = \frac{-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})}{h^2} + \frac{h^2}{24}\left(u^{(4)}(x_i) + u^{(4)}(\xi_i)\right)$$

thus $(-u''(x_i) = f(x_i)\ i = 1, \dots, N)$

$$\frac{-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})}{h^2} = f(x_i) + r_i \text{ with } |r_i| \leq \frac{1}{12} \max_{0 \leq x \leq 1} \left|u^{(4)}(x)\right| h^2$$

The finite diff scheme is obtained by replacing $u(x_i)$ by u_i and forgetting the r_i

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i) \quad i = 1, \dots, N$$

with $u_0 = 0$ and $u_{N+1} = 0$

This numerical scheme corresponds to the following linear system.

where A

$A\vec{v} = \vec{0}$

Questi

Thm

If u

$$A\vec{u} = \vec{f}$$

where $A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & (0) \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ (0) & & & -1 & 2 \end{pmatrix}$ tridiagonal matrix $N \times N$

$\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ $\vec{f} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix}$

Lemma: A is symmetric positive definite, therefore $\forall \vec{f} \in \mathbb{R}^N \; \exists! \, \vec{u} \in \mathbb{R}^N$ s.t. $A\vec{u} = \vec{f}$

Proof: $\forall \vec{v} \in \mathbb{R}^N \quad \vec{v}^T A \vec{v}$

$= 2v_1^2 - 2v_1v_2 + 2v_2^2 - 2v_2v_3 + \cdots - 2v_{N-1}v_N + 2v_N^2$

$= v_1^2 + (v_1 - v_2)^2 + \cdots + (v_{N-1} - v_N)^2 + v_N^2 \geq 0$

$= 0 \Leftrightarrow \vec{v} = \vec{0}$

$A\vec{v} = \vec{0} \Rightarrow \vec{v}^T A \vec{v} = 0 \Leftrightarrow \vec{v} = \vec{0}$ $\ker A = \vec{0}$

Question: when $N \to \infty$ $(h \to 0)$ does $u_i \to u(x_i)$?

Thm ("convergence in the $\|\cdot\|_\infty$ norm")

If $u \in \mathcal{C}^4[0,1]$ $\exists C > 0$ $\forall 0 < h \leq 1$

$$\max_{1 \leq i \leq N} |u(x_i) - u_i| \leq Ch^2$$

Interpretation: $\|\vec{v}\|_\infty = \max_{1 \leq i \leq N} |v_i|$

C depends on u, but not on h (or N)

Exp: choose $u \in \mathcal{C}^4[0,1]$, compute f, solve $A\vec{u} = \vec{f}$ and check

$\max_{1 \leq i \leq N} |u(x_i) - u_i|$ is divided by $2^2 = 4$ each time h is divided by 2

Proof: $\vec{U} = \begin{pmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix}$ $\quad A\vec{U} = \vec{f} + \vec{r}$ $\quad |r_i| \le \frac{1}{12} \max_{0 \le x \le 1} |u^{(4)}(x)| h^2$

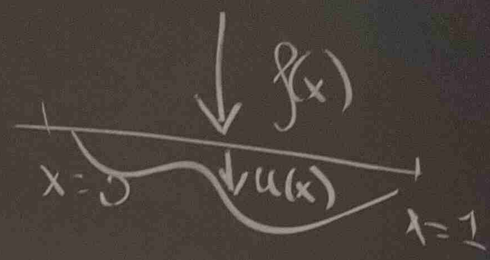
$$A(\vec{U} - \vec{u}) = \vec{r}$$

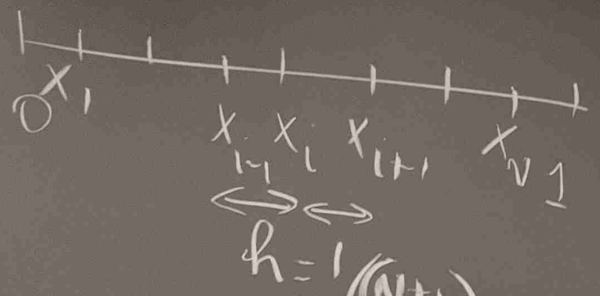
Lemma: $\forall \vec{g} \in \mathbb{R}^N$ let $\vec{v} \in \mathbb{R}^N$ s.t. $A\vec{v} = \vec{g}$ then $\|\vec{v}\|_\infty \le \frac{1}{8} \|\vec{g}\|_\infty$

thus $\max_{1 \le i \le N} |u(x_i) - u_i| = \|\vec{U} - \vec{u}\|_\infty \le \frac{1}{8} \|\vec{r}\|_\infty \le \frac{1}{12} \frac{1}{8} \max_{0 \le x \le 1} |u^{(4)}(x)| h^2$

Elliptic pbms $d=1$

$$\begin{cases} -u''(x) = f(x) & 0<x<1 \\ u(0)=0 \quad u(1)=0 \end{cases}$$





$x_i = ih \quad i = 0, 1, \dots, N+1$

u_i approx $u(x_i)$ $i = 1, \dots, N$

Finite differences (Taylor exp.)

Scheme $A\bar{u} = \bar{f}$ $\quad \bar{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ $\bar{f} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix}$

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & (0) \\ -1 & \ddots & \ddots \\ & \ddots & \ddots & -1 \\ (0) & & -1 & 2 \end{pmatrix}$$

Exact sol $\bar{U} = \begin{pmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix}$ $\quad A\bar{U} = \bar{f} + \bar{r}$

$$|r_i| \leq \frac{1}{12} \max_{0<x<1} |u^{(4)}(x)| \, h^2 \quad \text{if } u \in C^4[0,1]$$

$\forall u \in \mathcal{C}^4[0,1]\ \exists C>0\ \forall\ 0<h\leq 1$

$$\underbrace{\|\vec{U}-\vec{u}\|_\infty}_{} \leq C h^2$$

$$= \max_{1\leq i\leq N} |u(x_i)-u_i|$$

Proof $A(\vec{U}-\vec{u}) = \vec{\tau}$

lemma: $\forall \vec{g}\in\mathbb{R}^N\ \exists!\ \vec{v}\ A\vec{v}=\vec{g}\ \ \|v\|_\infty \leq \frac{1}{8}\|g\|_\infty$

$$\|\vec{U}-\vec{u}\| \leq \frac{1}{8}\|h\|_\infty \leq \underbrace{\frac{1}{8}\frac{1}{12}\max_{0\leq x\leq 1}|u^{(4)}(x)|}_{C} h^2$$

Proof of the lemma:

- Assume that $\exists\,\vec{w}$ such that $A\vec{w}=\vec{1}$
  (intuition: $-w''(x)=1$ $w(0)=0=w(1)$ $w(x)=\frac{x(1-x)}{2}$)
- A is monotone: $A\vec{z}\geqslant\vec{0} \Rightarrow \vec{z}\geqslant\vec{0}$
  (intuition $f(x)\geqslant 0 \Rightarrow u(x)\geqslant 0$)

by def of $\|\cdot\|_\infty$ $\forall \vec{g}\in\mathbb{R}^N$

$$-\|g\|_\infty \leq g_i \leq +\|g\|_\infty \quad i=1,\dots,N$$

that is $-\vec{1}\|g\|_\infty \leq \vec{g} \leq \vec{1}\|g\|_\infty$

Therefore

$$A(\vec{w}\|g\|_\infty) = \vec{1}\|g\|_\infty$$

$$A\ \vec{\tau}\ = \vec{g}$$

$$A(\vec{w}\|g\|_\infty + \vec{v}) = \vec{1}\|g\|_\infty + \vec{g} \geqslant \vec{0}$$

$$A(\ \ "\ \ -\ \ ") = \ \ "\ \ -\ \ \ \geqslant \vec{0}$$

Since A is monotone

$$-\vec{w}\|g\|_\infty \leq \vec{\tau} \leq \vec{w}\|\vec{g}\|_\infty$$

$$-w_i\|g\|_\infty \leq v_i \leq w_i\|g\|_\infty$$

$$\|\vec{v}\|_\infty \leqslant \|\vec{w}\|_\infty \|\vec{g}\|_\infty$$

$\leq \frac{1}{8}\|h\|_\infty \leq \frac{1}{8}\frac{1}{12}\max_{0\leq x\leq 1}$

C

he lemma,

at $\exists\, \vec{w}$ such that $A\vec{w} = \vec{1}$

on $-w''(x) = 1$ $\quad w(x) = \frac{x(1-x)}{2}$
$w(0) = 0 = w(1)$

orme, $A\vec{z} \geqslant \vec{0} \Rightarrow \vec{z} \geqslant \vec{0}$
$f(x) \geqslant 0 \Rightarrow u(x) \geqslant 0$)

$\|\vec{g}\|_\infty + \vec{g} \geqslant \vec{0}$

$\geqslant \vec{0}$

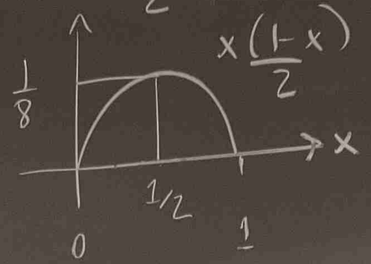
$\|\vec{g}\|_\infty$

$\|\vec{g}\|_\infty$

$\|\vec{w}\|_\infty \|\vec{g}\|_\infty$

What is $\vec{w}$? $w_i = \frac{x_i(1-x_i)}{2}$ check (intuition $w(x_i) = w_i = x_i\frac{(1-x_i)}{2}$)

$\|\vec{w}\|_\infty$?



$\|\vec{w}\|_\infty \leq \frac{1}{8}$

Finally $\|\vec{v}\|_\infty \leq \frac{1}{8}\|\vec{g}\|_\infty$

Proof that A is monotone

if $\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}\begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ then $\begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Consider index k such that $z_k \leq z_i \quad i = 1, \dots, N$
$= \min_{1\leq i\leq N}(z_i)$

| it suffices to prove that $z_k \geqslant 0$

If $k=1$ first row: $2z_1 - z_2 \geqslant 0$
$2z_1 \geqslant z_2$ but $z_2 \geqslant z_1$ thus $2z_1 \geqslant z_2 \geqslant z_1$
$z_1 \geqslant 0$

Same if $k=N$ last row $z_N \geqslant 0$

If $2 \leq k \leq N-1$ row k $\quad -z_{k-1} + 2z_k - z_{k+1} \geq 0$

that is $2z_k \geq z_{k-1} + z_{k+1} \geq z_{k-1} + z_k$

$z_k \geq z_{k-1}$ but $z_k \leq z_{k-1}$ thus $z_k = z_{k-1}$

repeat until reach index 1 ; $z_k = z_{k-1} = z_{k-2} = \dots = z_1$

but if $k=1$ then $z_1 \geq 0$ $\quad z_k \geq 0$

Remark ($A\vec{z} \geq 0 \Rightarrow \vec{z} \geq \vec{0}$) is true for instance if

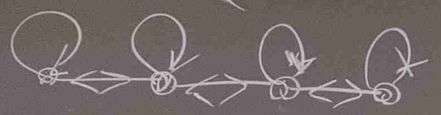
$a_{ii} > 0$ $\quad a_{ij} \leq 0$ $\quad j \neq i$ $\quad a_{ii} \geqslant \sum_{j \neq i} |a_{ij}|$ (diag. dom. ratos)

strictly, on at least one line,

and A irreducible (graph A strongly connected)

Ex:

$$\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 3 & -1 & & \\ -2 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -2 & 3 \end{pmatrix}$$



$$\begin{cases} -u''(x) = f(x) & 0<x<1 \\ u(0) = 0 \quad u'(1) = 0 \end{cases}$$

false $\begin{pmatrix} 1 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{pmatrix}$ singular $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ kernel

Thm (convergence in the $\|\cdot\|_2$ norm)
If $u \in C^4[0,1]$ $\exists C>0$ $\forall 0<h\leq 1$

$$\underbrace{\left(\sum_{i=1}^{N} (u(x_i)-u_i)^2\right)^{1/2}}_{\|\bar{U}-\bar{u}\|_2} \leq C h^{3/2}$$

Proof: Use the previous thm:

$$\left(\sum_{i=1}^{N} (u(x_i)-u_i)^2\right)^{1/2} \leq \left(N \max_{1\leq i\leq N} (u(x_i)-u_i)^2\right)^{1/2} \leq \sqrt{N}\, C h^2 \leq C h^{3/2}$$

but $N \leq N+1 = \frac{1}{h}$

Other proof with the eigenvalues of A
Since A is sym positive definite there exists an orthonormal basis of eigenvectors and the eigenvalues are strictly positive

$A\bar{\varphi}_i = \lambda_i \bar{\varphi}_i$

$\bar{\varphi}_i^T \bar{\varphi}_j = \delta_{ij} \quad i,j = 1,\dots,N$

$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$

Then $\forall v \in \mathbb{R}^N$

$$\lambda_1 \|\bar{v}\|_2^2 \leq \bar{v}^T A \bar{v} \leq \lambda_N \|\bar{v}\|_2^2$$

(expand $\bar{v}$ in the basis of eigenvectors $\bar{v} = \sum_{j=1}^{N} v_i \bar{\varphi}_i$)

We have

$$\lambda_1 \|\vec{U}-\vec{u}\|_2^2 \le (\vec{U}-\vec{u})^T A (\vec{U}-\vec{u}) = (\vec{U}-\vec{u})^T \vec{r} \le \|\vec{U}-\vec{u}\|_2 \, \|\vec{r}\|_2$$

$$\|\vec{U}-\vec{u}\|_2 \le \frac{1}{\lambda_1}\|\vec{r}\|_2 \le \frac{1}{\lambda_1}\sqrt{N}\|\vec{r}\|_\infty \le \frac{Ch^{3/2}}{\lambda_1} \left(C = \frac{1}{12}\max_{0\le x\le 1}|u^{(4)}(x)| \right)$$

What is λ_1?

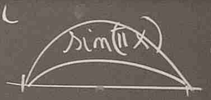
Explicit formula $\lambda_i = \left(\frac{2-2\cos(ih)}{h^2}\right) = 2\left(1-\cos\frac{i\pi}{N+1}\right)(N+1)^2$ $\quad i = 1, \dots, N$

$$\lambda_1 = 2\left(1-\cos\frac{\pi}{N+1}\right)(N+1)^2$$

(intuition $\quad -\varphi''(x) = \lambda\varphi(x) \quad \varphi(0) = 0 = \varphi(1)$

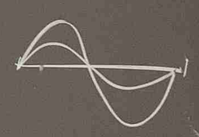
$\varphi(x) = \sin(i\pi x), \lambda_i = i^2\pi^2$

$\lambda_1 = \pi^2$



$N=2 \quad A = 3\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

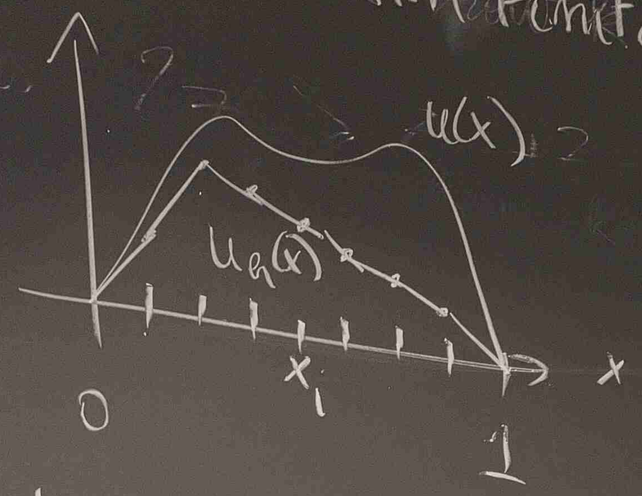
$\lambda_1 = 9 \quad \lambda_2 = 27$ (use eig with matlab)



$N \to \infty \quad \lambda_1 \to \pi^2 \quad \lambda_N \to +\infty$

Therefore $\lambda_1 \ge 9$ (can be proved)

Remark: Link with Finite elements



$u_h(x_i) = u_i$ $(A\vec{u} = \vec{f})$

same linear system

$$\|u - u_h\|_{L^2(0,1)} = \left(\int_0^1 (u(x) - u_h(x))^2 dx \right)^{1/2} \leq C h^2$$

$$\|u - u_h\|_{L^2(0,1)} = \left(\sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} (u(x) - u_h(x))^2 dx \right)^{1/2}$$

$$\sim \left(\sum_{i=0}^{N} h (u(x_i) - u_i)^2 \right)^{1/2} = \sqrt{h} \, \|\vec{U} - \vec{u}\|_2$$

Thm (Convergence in the $(\vec{v}^T A \vec{v})^{1/2}$ norm)

$\forall u \in \mathcal{C}^4[0,1] \; \exists C > 0 \; \forall \alpha < h \leq 1 \quad \left((\vec{U} - \vec{u})^T A (\vec{U} - \vec{u}) \right)^{1/2} \leq C h^{3/2}$

Linked with finite elements

$$\|u'-u_h'\|_{L^2(0,1)} = \left(\int_0^1 (u'(x)-u_h'(x))^2 dx\right)^{1/2} \leq Ch$$

Proof:

$$(\vec{U}-\vec{u})^T A(\vec{U}-\vec{u}) = (\vec{U}-\vec{u})^T \vec{r} \leq \|\vec{U}-\vec{u}\|_2 \|\vec{r}\|_2$$

$$\leq \frac{1}{\sqrt{\lambda}} \left((\vec{U}-\vec{u})^T A(\vec{U}-\vec{u})\right)^{1/2} Ch^{3/2}$$

$$\left((\vec{U}-\vec{u})^T A(\vec{U}-\vec{u})\right)^{1/2} \leq \frac{1}{\sqrt{\lambda}} Ch^{3/2}$$

$\frac{u(x_i)-u(x_{i-1})}{h}$

Extensions

$$\star \begin{cases} -u''(x)+c(x)u(x) = f(x) & 0<x<1 \\ u(0)=0 \quad u(1)=0 \end{cases}$$

where $c(x) \geq 0$

Steps of the lemma $A\vec{v}=\vec{g}$ $\|\vec{r}\|_\infty \leq C\|\vec{g}\|_\infty$

Find a $\vec{w}$ st $A\vec{w} \geqslant \vec{1}$

A monotone $(A\vec{z} \geqslant 0) \Rightarrow (\vec{z} \geqslant \vec{0})$

true since positive extra term in A

scheme $\left\{ -\frac{u_{i-1}-2u_i+u_{i+1}}{h^2} + c(x_i)u_i = f(x_i) \right.$ $i=1,\dots,N$

$u_0=0=u_{N+1}$

$A\vec{u}=\vec{f}$

$$A = \begin{pmatrix} \frac{2}{h^2}+c(x_1) & -\frac{1}{h^2} & & \\ -\frac{1}{h^2} & \ddots & \ddots & \\ & \ddots & \ddots & -\frac{1}{h^2} \\ & & -\frac{1}{h^2} & \frac{2}{h^2}+c(x_N) \end{pmatrix}$$

We have

$$\lambda_1 \|\vec{v}-\vec{u}\|_2^2 \le (\vec{v}-\vec{u})^T A (\vec{v}-\vec{u}) = (\vec{v}-\vec{u})^T \vec{r} \le \|\vec{v}-\vec{u}\|_2 \, \|\vec{r}\|_2$$

$$\|\vec{v}-\vec{u}\|_2 \le \frac{1}{\lambda_1}\|\vec{r}\|_2 \le \frac{1}{\lambda_1}\sqrt{N}\|\vec{r}\|_\infty \le \frac{Ch^{3/2}}{\lambda_1} \qquad \left(C = \frac{1}{12}\max_{0\le x\le 1}|u^{(4)}(x)|\right)$$

What is λ_1 ?

Explicit formula $\lambda_i = \left(\frac{2-2\cos(i\pi h)}{h^2}\right) = 2\left(1-\cos\frac{i\pi}{N+1}\right)(N+1)^2 \qquad i = 1, \dots, N$

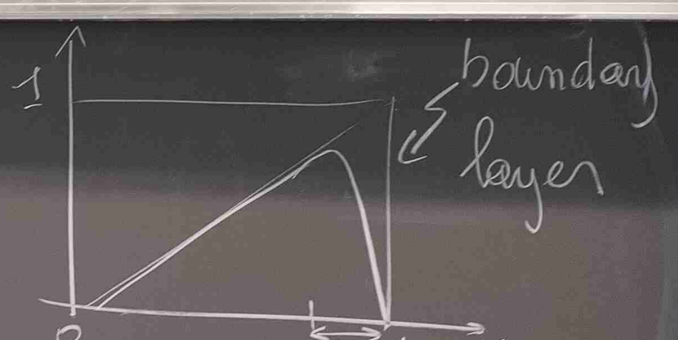
$$\lambda_1 = 2\left(1-\cos\frac{\pi}{N+1}\right)(N+1)^2$$

$$* \begin{cases} -u''(x) = f(x) & 0<x<1 \\ u(0)=0 \quad u'(1)=0 \end{cases}$$

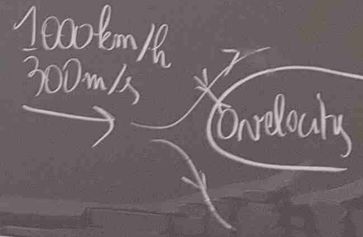
Exercice
Find $\vec{w}$ st $A\vec{w} \ge \vec{1}$
A monotone

* Advection-diffusion $\varepsilon > 0$

$$\begin{cases} -\varepsilon u''(x) + u'(x) = 1 \\ u(0)=0 \quad u(1)=0 \end{cases}$$



$$u(x) = x - \frac{1-e^{x/\varepsilon}}{1-e^{1/\varepsilon}}$$

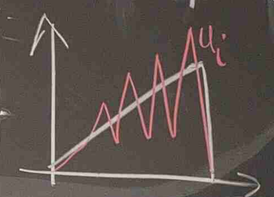


Centered scheme:

$$\begin{cases} -\varepsilon \dfrac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \dfrac{u_{i+1} - u_{i-1}}{2h} = 1 \quad i = 1, \dots, N \\ u_0 = 0 \qquad u_{N+1} = 0 \end{cases}$$

$A\vec{u} = \vec{f}$

Exp: set $h = 0{,}01$ $\varepsilon = 0{,}1$ / $0{,}01$ / $0{,}001$ / $0{,}0001$



$$A = \begin{pmatrix} & \ddots & \ddots & \ddots & \\ & -\frac{\varepsilon}{h^2} - \frac{1}{2h} & \frac{2\varepsilon}{h^2} & -\frac{\varepsilon}{h^2} + \frac{1}{2h} & \\ & \ddots & \ddots & \ddots & \end{pmatrix}$$

$A(\vec{v} - \vec{u}) = \vec{\pi} \qquad \|\vec{\pi}\|_\infty \leq Ch^2$

$A\vec{w} \geq \vec{1}$ and $A\vec{z} \geq \vec{0} \Rightarrow \vec{z} \geq \vec{0}$

Stability condition

$$-\frac{\varepsilon}{h^2} + \frac{1}{2h} < 0$$

$$\frac{\varepsilon}{h^2} - \frac{1}{2h} \geq 0$$

$$\frac{\varepsilon}{h} \geq \frac{1}{2}$$

$$\boxed{h \leq 2\varepsilon}$$

Proof:

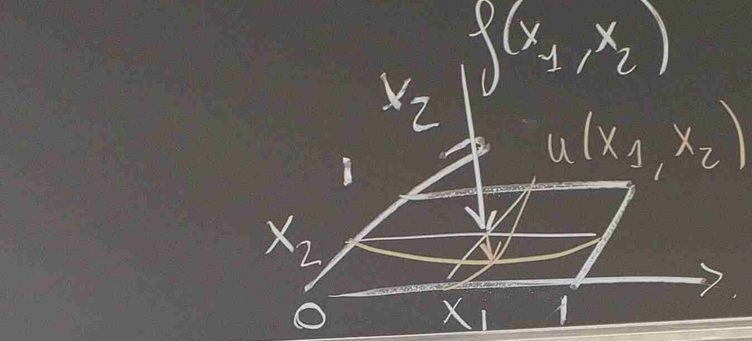
part 2 FD pde's

1. Elliptic

$d = 1 \quad -u''(x) = f(x) \quad 0 < x < 1$

$d = 2, 3 \longrightarrow$ solve the linear system $A\bar{u} = \bar{f}$

d = 2



Given $f : \Omega =]0,1[^2 \longrightarrow \mathbb{R}$ find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} -\Delta u(x_1, x_2) = f(x_1, x_2) & \forall (x_1, x_2) \in \Omega \\ u(x_1, x_2) = 0 & \in \partial\Omega \end{cases}$$

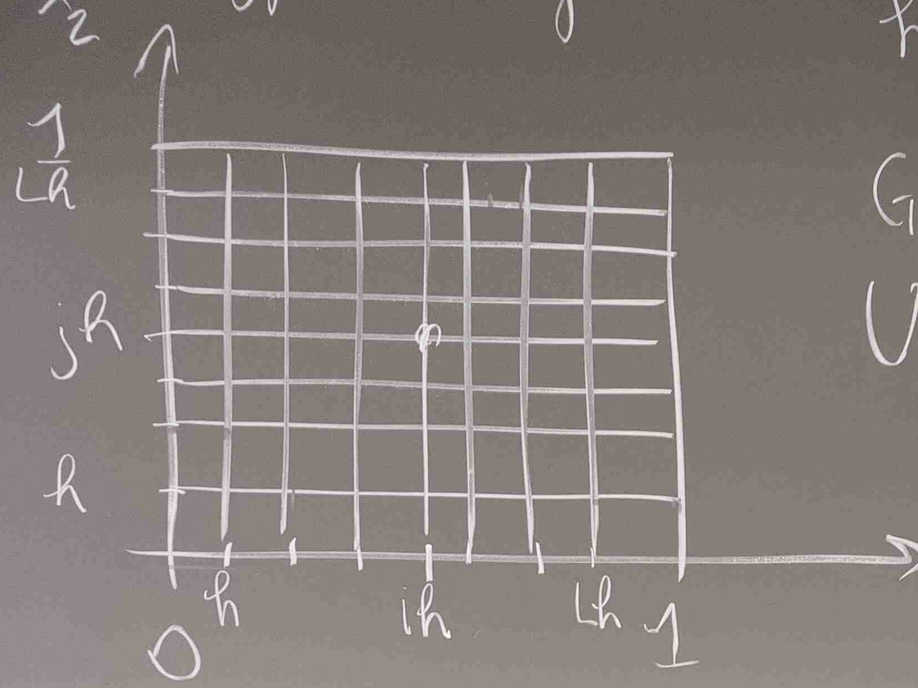
$\Delta u = \operatorname{div}(\operatorname{grad} u)$

$= \vec{\nabla} \cdot \vec{\nabla} u$

$= \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$

$\vec{\nabla} u = \begin{pmatrix} \frac{\partial}{\partial x_1} u \\ \frac{\partial}{\partial x_2} u \end{pmatrix}$

Finite difference grid



$$h = \frac{1}{L+1}$$

Goal to compute

$U_{i,j}$ approx of $u(ih, jh)$

$i,j = 1, \dots, L$

Write the pde at ih, jh

$$-\Delta u(ih, jh) = f(ih, jh) \qquad i,j = 1, \dots, L$$

FD approx of $\frac{\partial^2 u}{\partial x_1^2}(ih, jh)$ and $\frac{\partial^2 u}{\partial x_2^2}$ ——

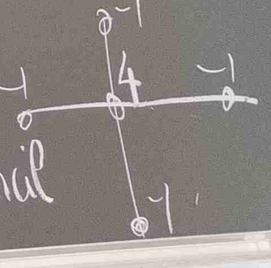
$$\frac{\partial^2 u}{\partial x_1^2}(ih, jh) = \frac{u((i-1)h, jh) - 2u(ih, jh) + u((i+1)h, jh)}{h^2} + O(h^2) \quad \text{if } u \in \mathcal{C}^4([0,1]^2)$$

$$\frac{\partial^2 u}{\partial x_2^2}(ih, jh) = \frac{u(ih, (j-1)h) - 2u(ih, jh) + u(ih, (j+1)h)}{h^2}$$

The scheme is obtained replacing $u(ih, jh)$ by $U_{i,j}$ and forgetting the $O(h^2)$ terms

$$\frac{-U_{i-1,j} - U_{i,j-1} + 4U_{i,j} - U_{i+1,j} - U_{i,j+1}}{h^2} = f(ih, jh) \qquad i,j = 1, \dots, L$$

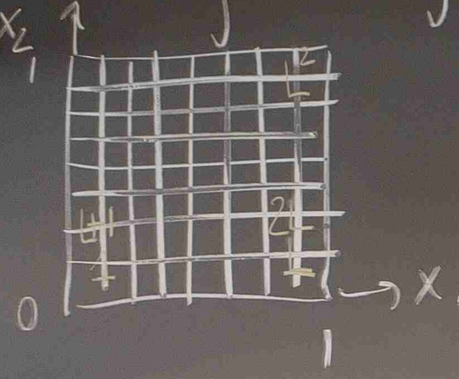
5 points stencil

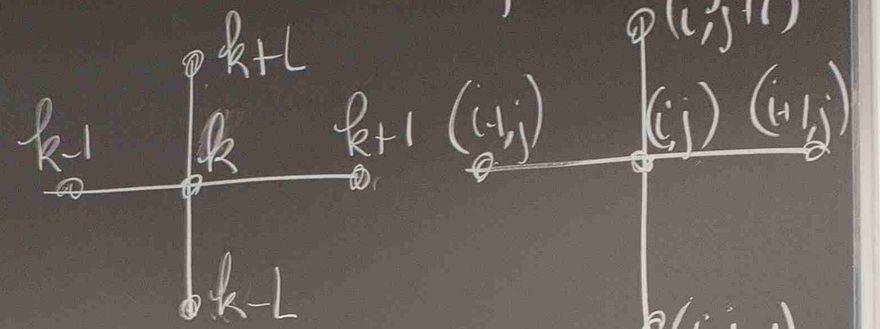


with the boundary conditions $U_{i,j} = 0$ if $i = 0$ or $i = L+1$

or $j = 0$ or $j = L+1$

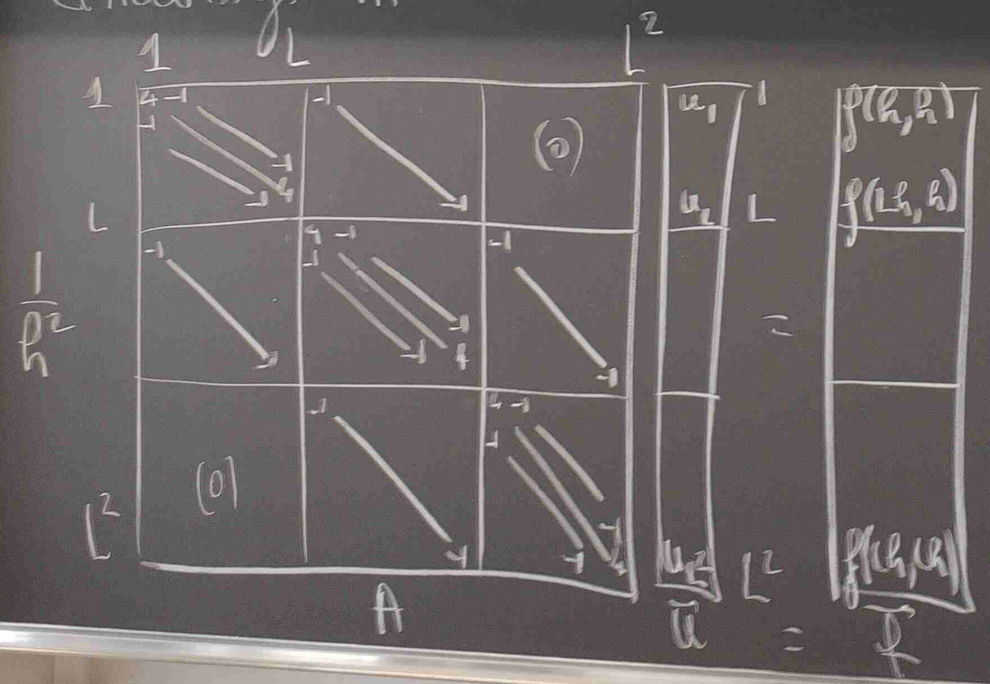
Numering row by row





$U_{i,j}$ is denoted as u_k

Linear system to be solved



matrix $L^2 \times L^2$

with 5 diagonals

bloc matrix

It can be shown that if $u \in C^4([0,1]^2)$ then

$\exists C > 0 \quad \forall 0 < h \leq \frac{1}{2}$

$$\max_{1 \leq i,j \leq L} |u(ih, jh) - U_{i,j}| \leq Ch^2$$

Implementation of the matrix A with matlab
octave
numpy

The matrix A can be written as

$$A = \frac{1}{h^2}\left(D_L \otimes I_L + I_L \otimes D_L\right)$$

where $\otimes$ is the kronecker product (kron matlab)

$$D_L = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_L \qquad I_L = \begin{pmatrix} 1 & & & & (0) \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ (0) & & & & 1 \end{pmatrix}_L$$

$B_1 C \in \mathbb{R}^{L \times L}$ $\quad B \otimes C \in \mathbb{R}^{L^2 \times L^2}$

$$B \otimes C = \begin{pmatrix} B_{11}C & \cdots & B_{1L}C \\ \vdots & \ddots & \vdots \\ B_{L1}C & \cdots & B_{LL}C \end{pmatrix}$$

$$A = \frac{1}{h^2} \left(\underbrace{\begin{pmatrix} 2I_L & -I_L & \\ -I_L & 2I_L & -I_L \\ & -I_L & 2I_L \end{pmatrix}}_{D_L \otimes I_L} + \underbrace{\begin{pmatrix} D_L & & \\ & D_L & \\ & & D_L \end{pmatrix}}_{I_L \otimes D_L} \right)$$

Memory needed to store A?

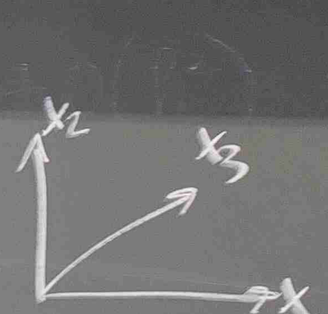
CPU ——— solve $A\bar{u} = \bar{f}$

Complexity

2 possible class of methods to solve $A\bar{u} = \bar{f}$

- Direct methods : Cholesky decomp.
- Iterative —— : gradient, conjugate gradient

$d = 3$





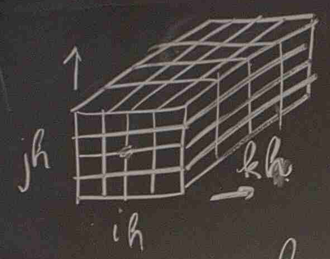
$\Omega =]0,1[^3$

Given $f : \Omega \to \mathbb{R}$ find $u : \Omega \to \mathbb{R}$ s.t

$$\begin{cases} -\Delta u(x_1, x_2, x_3) = f(x_1, x_2, x_3) & \forall (x_1 x_2 x_3) \in \Omega \\ u(x_1, x_2, x_3) = 0 & \text{——} \in \partial\Omega \end{cases}$$

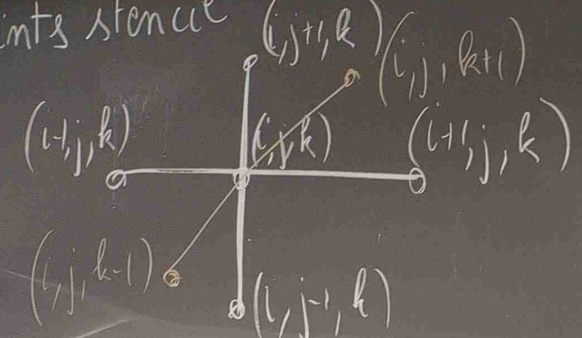
$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

$h = \frac{1}{L+1}$



$U_{i,j,k}$ approx of $u(ih, jh, kh)$ $\quad i,j,k = \overline{1,L}$

7 points stencil



The FD scheme is

$$\frac{-U_{i-1,j,k} - U_{i,j-1,k} - U_{i,j,k-1} + 6U_{i,j,k} - U_{i+1,j,k} - U_{i,j+1,k} - U_{i,j,k+1}}{h^2} = f(ih, jh, kh)$$

$i,j,k = \overline{1,\dots,L}$

$U_{i,j,k} = 0$ if $i = 0$ or $i = L+1$
$j = 0$ $\quad j = L+1$
$k = 0$ $\quad k = L+1$

- Numbering row by row in each slice

$A\bar{u} = f \qquad \bar{u} \in \mathbb{R}^{L^3} \qquad A \in \mathbb{R}^{L^3 \times L^3}$

Implementation with matlab (use spy to see non zero coeff)

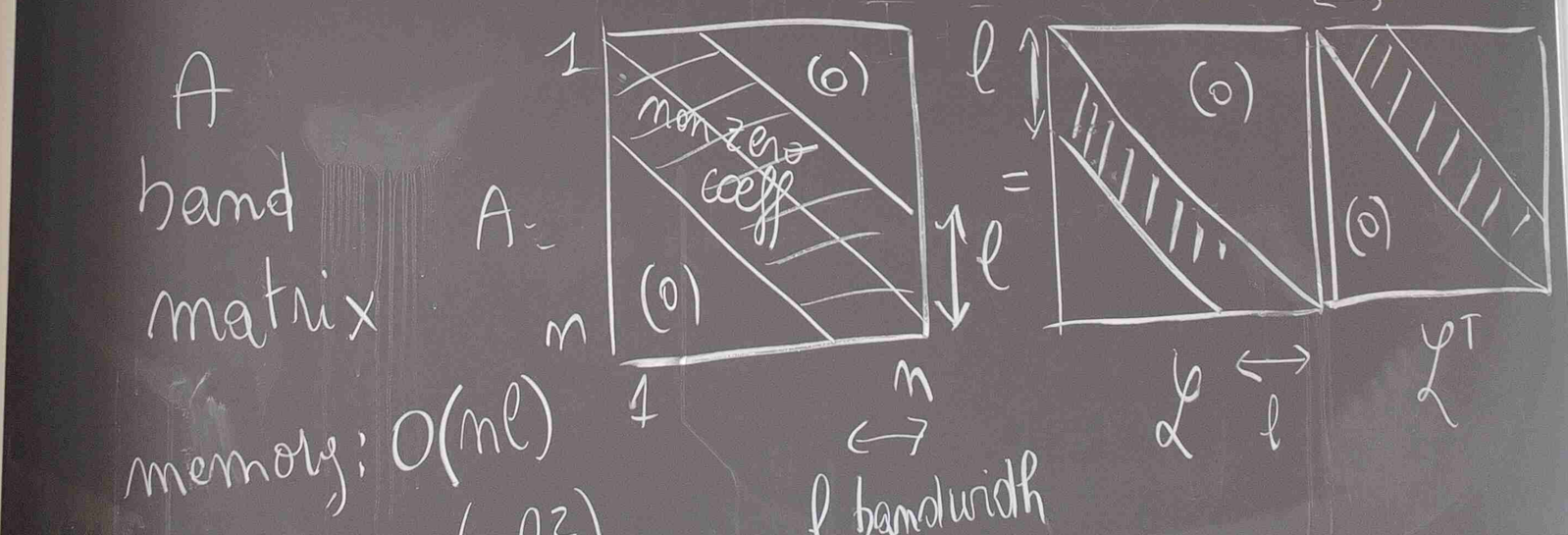
$$A = \frac{1}{h^2}\left(I_L \otimes I_L \otimes \gamma_L + I_L \otimes \gamma_L \otimes I_L + \gamma_L \otimes I_L \otimes I_L \right)$$

Cholesky dec $A = \mathcal{L}\mathcal{L}^T$: fill the band of $\mathcal{L}$

A band matrix

memory : $O(nl)$

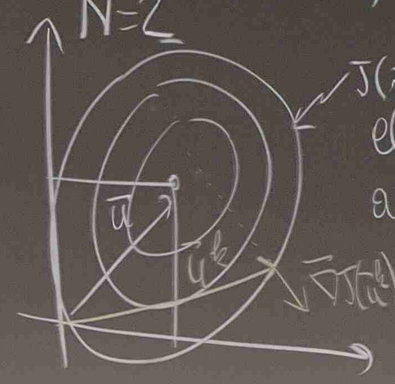
CPU : $O(nl^2)$



gradient method to solve $A\vec{u} = \vec{f}$ (A sym. pos. def.)

$$J(\vec{u}) \leq J(\vec{v}) \quad \forall \vec{v} \in \mathbb{R}^N \qquad (N = L^d \; d = 1,2,3)$$

$$J(\vec{v}) = \frac{1}{2}\vec{v}^T A \vec{v} - \vec{f}^T \vec{v}$$



Choose $\vec{u}^0 \in \mathbb{R}^N$

$k = 1, 2, \ldots$

$\vec{r}^k = \vec{f} - A\vec{u}^k$

$\vec{u}^{k+1} = \vec{u}^k + \alpha^k \vec{r}^k$

α^k s.t. $J(\vec{u}^k + \alpha^k \vec{r}^k)$

$\leq J(\vec{u}^k + \alpha \vec{r}^k) \quad \forall \alpha \in \mathbb{R}$

$\ell(\alpha) \quad \ell : \mathbb{R} \to \mathbb{R}$

$$\ell'(\alpha) = \left(A(\vec{u}^k + \alpha \vec{r}^k) - \vec{f}\right)^T \vec{r}^k$$

$$\ell'(\alpha^k) = 0 \qquad \alpha^k = \frac{\vec{r}^{kT}\vec{r}^k}{\vec{r}^{kT} A \vec{r}^k}$$

number of iterations?

Can be proved: $\|\vec{u}^{k+1} - \vec{u}\|_A \leq \|\vec{u}^k - \vec{u}\| \dfrac{K(A)-1}{K(A)+1}$

$\left(\|\vec{v}\|_A^2 = \vec{v}^T A \vec{v}\right)$

$$K(A) = \frac{\lambda_N}{\lambda_1} \qquad A\vec{\varphi}_i = \lambda_i \vec{\varphi}_i \qquad \vec{\varphi}_i^T \vec{\varphi}_j = \delta_{ij}$$

$0 < \lambda_1 \leq \ldots \leq \lambda_N$

Can be proved:

$K(A) = O(L^2)$ $\quad \left(-\varphi_i''(x) = \lambda_i \varphi_i(x) \quad 0<x<1 \quad \varphi_i(0) = \varphi_i(1) = 0 \quad \varphi_i(x) = \sin(i\pi x) \quad \lambda_i = i^2\pi^2\right)$

$$\frac{K(A)-1}{K(A)+1} = \frac{1 - \frac{1}{K(A)}}{1 + \frac{1}{K(A)}} = 1 - O\left(\frac{1}{L^2}\right)$$

number of iter?

k s.t. $\|\vec{u}^k - \vec{u}\|_A = \varepsilon \|\vec{u}^0 - \vec{u}\|_A$ ε small 10^{-6}

$$\varepsilon \|\vec{u}^0 - \vec{u}\| = \|\vec{u}^k - \vec{u}\|_A \leq \|\vec{u}^0 - \vec{u}\|_A \left(\frac{K(A)-1}{K(A)+1}\right)^k$$

$$\varepsilon \leq \left(\frac{K(A)-1}{K(A)+1}\right)^k = \left(1 - O\left(\frac{1}{L^2}\right)\right)^k = 1 - O\left(\frac{k}{L^2}\right)$$

$$\frac{k}{L^2} = O(1+\varepsilon) \quad \underline{k = O(L^2)}$$

Conjugate gradient $\vec{u}^0$, $\vec{r}^0 = \vec{f} - A\vec{u}^0 = \vec{d}^0$

$k = 1, 2, \ldots$

$\vec{u}^{k+1} = \vec{u}^k + \alpha^k \vec{d}^k$ $\quad \alpha^k$ st $\vec{J}(\vec{u}^k + \alpha^k \vec{d}^k) \leq \vec{J}(\vec{u}^k + \alpha \vec{d}^k)$ $\forall \alpha \in \mathbb{R}$

$\vec{r}^{k+1} = \vec{f} - A\vec{u}^{k+1}$ $\quad \alpha^k = \dfrac{\vec{d}^{kT} \vec{r}^k}{\vec{d}^{kT} A \vec{d}^k}$

$\vec{d}^{k+1} = \vec{r}^{k+1} + \beta^{k+1} \vec{d}^k$ $\quad \beta^{k+1}$ st $\vec{d}^{k+1\,T} A \vec{d}^k = 0$

Can be proved $\|\vec{u}^{k+1} - \vec{u}\|_A \leq \|\vec{u}^k - \vec{u}\|_A \dfrac{\sqrt{K(A)} - 1}{\sqrt{K(A)} + 1}$ therefore $k = O(L)$

$d=1,2,3$	Cholesky	Conj. grad
unknowns	$O(L^d)$	$O(L^d)$
memory $d=3$	$O(L^d L^{d-1})$ $O(L^5)$	$O(L^d)$ $O(L^3)$
CPU $d=3$	$O(L^d L^{2(d-1)})$ $O(L^7)$	$O(L^{d+1})$ $O(L^4)$

Part 2 FD

1 - Elliptic
2 - Parabolic problems : heat equation
3 - hyperbolic problems : transport, wave

2 - Heat eq 1d

Data : initial condition $u_0 : [0,1] \to \mathbb{R}$, $x \to u_0(x)$

source term $f : [0,1] \times [0,T] \to \mathbb{R}$, $(x,t) \to f(x,t)$

Unknown : temperature $u : [0,1] \times [0,T] \to \mathbb{R}$, $(x,t) \to u(x,t)$

such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) & 0<x<1 \quad 0<t<T \\ u(x,0) = u_0(x) & \\ u(0,t) = 0 \quad u(1,t) = 0 & \end{cases}$$

Remarks:

- Fourier series: if $u_0(x) = \sin(\pi x)$ and $f(x,t) = \sin(\pi x)$ then $u(x,t) = \sin(\pi x)e^{-\pi^2 t} + \frac{1}{\pi^2}(1 - e^{-\pi^2 t})\sin(\pi x)$

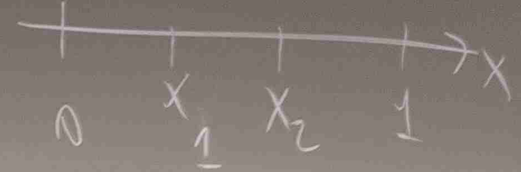
  (first term in the Fourier expansion)

- Maximum principle, positivity

If $u_0(x) \geq 0$ and $f(x,t) \geq 0$ then $u(x,t) \geq 0$

If $f(x,t) = 0$ then $\max_{0 \leq x \leq 1} |u(x,t)| \leq \max_{0 \leq x \leq 1} |u_0(x)|$

- The heat eq. corresponds to an energy cons. principle:



int. between $x = x_1$ and $x = x_2$

$$\frac{d}{dt}\underbrace{\int_{x_1}^{x_2} u(x,t)\,dx}_{\text{thermal energy}} - \underbrace{\left[\frac{\partial u}{\partial x}(x,t)\right]_{x=x_1}^{x=x_2}}_{\text{ingoing or outgoing heat flux}} = \underbrace{\int_{x_1}^{x_2} f(x,t)\,dx}_{\text{thermal energy by source}}$$

Variation over time

- A priori estimate: if the solution exists then it satisfies a certain a priori estimate; multiply by $u(x,t)$ and $\int_0^1 dx$

$$\int_0^1 \frac{\partial u}{\partial t}(x,t)\,u(x,t)\,dx - \int_0^1 \frac{\partial^2 u}{\partial x^2}(x,t)\,u(x,t)\,dx = \int_0^1 f(x,t)\,u(x,t)\,dx$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u(x,t))^2\,dx + \int_0^1 \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx - \left[\frac{\partial u}{\partial x}(x,t)\underbrace{u(x,t)}_{0}\right]_{x=0}^{x=1} = \int_0^1 f(x,t)\,u(x,t)\,dx$$

Use Cauchy-Schwarz ineq. $\int_0^1 f(x,t)\,u(x,t)\,dx \le \left(\int_0^1 f^2\,dx\right)^{1/2}\left(\int_0^1 u^2\,dx\right)^{1/2}$

and Poincaré ineq. $\int_0^1 (u(x,t))^2\,dx \le \int_0^1 \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx$

and $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ (a, b)

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u(x,t))^2\,dx + \frac{1}{2}\int_0^1 \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx \le \frac{1}{2}\int_0^1 (f(x,t))^2\,dx$$

Integrate between $t=0$ and $t=T$

$$\int_0^1 (u(x,T))^2\,dx + \int_0^T\int_0^1 \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx\,dt \le \int_0^1 (u_0(x))^2\,dx + \int_0^T\int_0^1 (f(x,t))^2\,dx\,dt$$

This is an a priori estimate, which implies uniqueness of u

If $f(x,t)=0$ u_0 discontinuous,

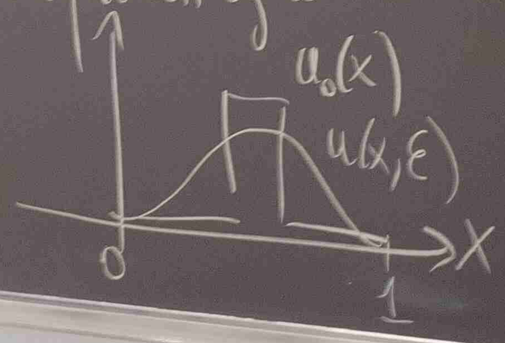
then $u \in C^\infty([0,1]\times[\varepsilon,T])$ for any $\varepsilon>0$

$$(uv)' = u'v + uv'$$

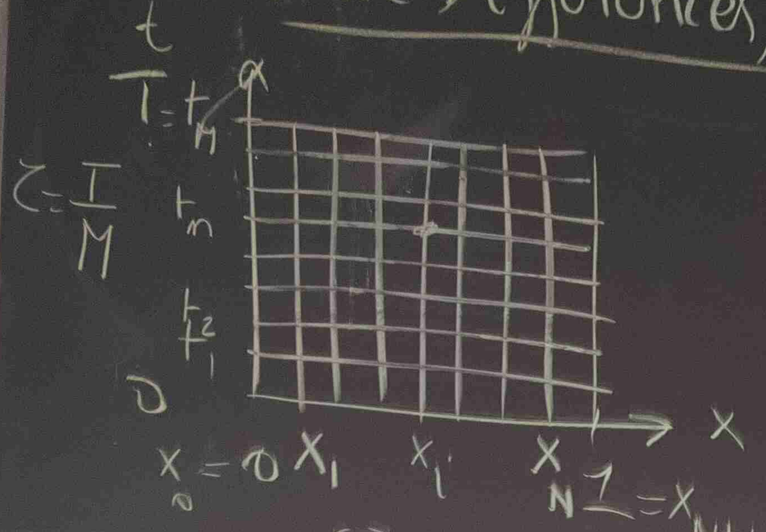
$$[uv] = \int u'v + \int uv'$$

$u \to u'$

$$[u'v] = \int u''v + \int u'v'$$



Finite Differences, implicit scheme



$\Leftrightarrow$

$h = \frac{1}{N+1}$

$x_i = ih \quad i = 0, 1, \dots, N, N+1$

$t_n = n\tau \quad n = 0, 1, \dots, M$

goal to compute u_i^n approx $u(x_i, t_m)$

Write the heat eq at (x_i, t_{m+1})

$$\frac{\partial u}{\partial t}(x_i, t_{m+1}) - \frac{\partial^2 u}{\partial x^2}(x_i, t_{m+1}) = f(x_i, t_{m+1})$$

Backward finite diff formula for $\frac{\partial u}{\partial t}$ (implicit Euler)

Centered

$$\frac{u(x_i, t_{m+1}) - u(x_i, t_m)}{\tau} - \frac{u(x_{i-1}, t_{m+1}) - 2u(x_i, t_{m+1}) + u(x_{i+1}, t_{m+1})}{h^2} \quad \frac{\partial^2 u}{\partial x^2}$$

$$= f(x_i, t_{m+1}) + r_i^{m+1} \qquad i = 1, \dots, N$$

with $|r_i^{m+1}| \leq \frac{h^2}{12} \max_{0 \leq x \leq 1} \left|\frac{\partial^4 u}{\partial x^4}(x, t)\right| + \frac{\tau}{2} \max_{0 \leq t \leq T} \left|\frac{\partial^2 u}{\partial t^2}(x, t)\right|$

$|r_i^{n+1}| \le C(h^2+\tau)$ if u is smoth enough

Scheme: replace $u(x_i,t_n)$ by u_i^n, forget r_i^{n+1}

$$\frac{u_i^{n+1}-u_i^n}{\tau} - \frac{u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}}{h^2} = f(x_i,t_{n+1}) \quad i=1,\dots,N$$

u_i^n are known (start from $u_i^0 = u(x_i)$)

$u_0^{n+1} = 0 \qquad u_{N+1} = 0$

Linear system to be solved at each time step

$$(I+\tau A)\vec{u}^{n+1} = \vec{u}^n + \tau \vec{f}^{n+1}$$

$$\vec{u}^n = \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix} \quad \vec{f}^n = \begin{pmatrix} f(x_1,t_n) \\ \vdots \\ f(x_N,t_n) \end{pmatrix} \quad A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & (0) \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ (0) & & -1 & 2 \end{pmatrix}$$

Note that the exact sol satisfies

$$(I+\tau A)\vec{U}^{n+1} = \vec{U}^n + \tau\vec{f}^{n+1} + \tau\vec{r}^{n+1}$$

where $\vec{U}^n = \begin{pmatrix} u(x_1,t_n) \\ \vdots \\ u(x_N,t_n) \end{pmatrix} \quad \vec{r}^n = \begin{pmatrix} r_1^n \\ \vdots \\ r_N^n \end{pmatrix}$

Stability and convergence in the $\|\cdot\|_\infty$ norm

$\|\vec{v}\|_\infty = \max_{1\le i\le N} |v_i| \quad \forall \vec{v}\in\mathbb{R}^N$

Lemma: stability (max. principle and positivity)

a. $\|\vec{u}^{n+1}\|_\infty \le \|\vec{u}^n\|_\infty + \tau\|f^{n+1}\|_\infty$

b. If $\vec{u}^n \ge \vec{0}$ and $\vec{f}^{n+1} \ge \vec{0}$ then $\vec{u}^{n+1} \ge \vec{0}$

Proof:

a. let k be such that $|u_k^{n+1}| \geq |u_i^{n+1}| \quad i = 1, \dots, N$

scheme at line k:

$$\left(\frac{1}{h^2} + \frac{2\tau}{h^2}\right) u_k^{n+1} = \frac{\tau}{h^2}\left(u_{k-1}^{n+1} + u_{k+1}^{n+1}\right) + u_k^n + \tau f(x_k, t_{n+1})$$

take the abs. value

$$\left(\frac{1}{h^2} + \frac{2\tau}{h^2}\right) |u_k^{n+1}| \leq \frac{\tau}{h^2}\left(|u_{k-1}^{n+1}| + |u_{k+1}^{n+1}|\right) + |u_k^n| + \tau |f(x_k, t_{n+1})|$$

$$\left(\frac{1}{h^2} + \frac{2\tau}{h^2}\right) |u_k^{n+1}| \leq \frac{\tau}{h^2}\left(|u_k^{n+1}| + |u_k^{n+1}|\right) + |u_k^n| + \tau |f(x_k, t_{n+1})|$$

(if $k=1$ or $k=N$ update the proof)

$$\|\vec{u}^{n+1}\|_\infty = |u_k^{n+1}| \leq |u_k^n| + \tau |f(x_k, t_{n+1})| \leq \|\vec{u}^n\|_\infty + \tau \|f^{n+1}\|_\infty$$

b. let k be such that $u_k^{n+1} \leq u_i^{n+1} \quad i = 1, \dots, N$

It suffices to prove that $u_k^{n+1} \geqslant 0$ whenever $\vec{u}^n \geqslant \vec{0}$ and $\vec{f}^{n+1} \geqslant \vec{0}$

Theorem: implicit scheme

$$\left(1+\frac{2\tau}{h^2}\right)u_k^{n+1} = \frac{\tau}{h^2}\left(u_{k-1}^{n+1} + u_{k+1}^{n+1}\right) + u_k^n + \tau f(x_k, t_{n+1})$$

$$\geqslant \frac{\tau}{h^2}\left(u_k^{n+1} + u_k^{n+1}\right) + u_k^n + \tau f(x_k, t_{n+1})$$

If $k \neq 1$ and $k \neq N$

$$u_k^{n+1} \geqslant u_k^n + \tau f(x_k, t_{n+1})$$

If u_k^n and $f(x_k, t_{n+1})$ are $\geqslant 0$ then $u_k^{n+1} \geqslant 0$

$k = 1$:

$$\left(1+\frac{2\tau}{h^2}\right)u_1^{n+1} = \frac{\tau}{h^2}\left(\underbrace{u_0^{n+1}}_{0} + u_2^{n+1}\right) + u_1^n + \tau f(x_1, t_{n+1})$$

$$\left(1+2\frac{\tau}{h^2}\right)u_1^{n+1} \geqslant \frac{\tau}{h^2}u_1^{n+1} + u_1^n + \tau f(x_1, t_{n+1})$$

$$\left(1+\frac{\tau}{h^2}\right)u_1^{n+1} \geqslant u_1^n + \tau f(x_1, t_{n+1})$$

If $u_1^n \geqslant 0$ and $f(x_1, t_{n+1}) \geqslant 0$ then $u_1^{n+1} \geqslant 0$

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Thm: Convergence in the $\|\cdot\|_\infty$ norm:

If u is $\mathcal{C}^4$ in space and $\mathcal{C}^2$ in time then there exist $D>0$ such that for all $h>0$ and $\tau>0$ we have

$$\max_{1\le i\le N} |u(x_i,t_M)-u_i^M| = \|\bar{U}^M-\bar{u}^M\|_\infty \le D(h^2+\tau).$$

with $\|r^{n+1}\|_\infty \le C(h^2+\tau)$ (C depends only on $\frac{\partial^4 u}{\partial x^4}$ and $\frac{\partial^2 u}{\partial t^2}$)

by induction

$$\|\bar{U}^M-\bar{u}^M\|_\infty \le \underbrace{\|\bar{U}^0-\bar{u}^0\|_\infty}_{0} + \tau\sum_{n=1}^{M}\|\bar{r}^n\|_\infty$$

$$\le \underbrace{M\tau}_{T}\, C(h^2+\tau) = \underbrace{CT}_{D}(h^2+\tau)$$

Interpretation: Choose u $\mathcal{C}^4$ sp $\mathcal{C}^2$ time, compute u_0, f, then the error at final time should be divided by 4 each time h is divided by 2 and τ by 4.

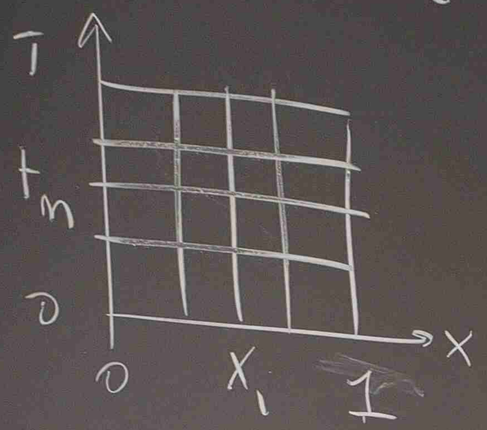
Proof: $(I+\tau A)(\bar{U}^{n+1}-\bar{u}^{n+1}) = \bar{U}^n-\bar{u}^n+\tau\bar{r}^{n+1}$

Lemma $\|\bar{U}^{n+1}-\bar{u}^{n+1}\|_\infty \le \|\bar{U}^n-\bar{u}^n\|_\infty + \tau\|\bar{r}^{n+1}\|_\infty$

Same with the explicit scheme
→ stability condition $\tau \le \frac{h^2}{2}$
but no linear to be solved

Same with the $\|\cdot\|_2$ norm
→ eigenvalues

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) & 0<x<1,\ 0<t\le T \\ u(0,t)=0 \quad u(1,t)=0 \\ u(x,0)=u_0(x) \end{cases}$$



Implicit $\frac{\partial u}{\partial t}(x_i,t_{n+1}) - \frac{\partial^2 u}{\partial x^2}(x_i,t_{n+1}) = f(x_i,t_{n+1})$

$$\frac{u_i^{n+1}-u_i^n}{\tau} - \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{h^2} = f(x_i,t_{n+1})$$

$$\frac{\vec{u}^{n+1}-\vec{u}^n}{\tau} + A\vec{u}^{n+1} = \vec{f}^{n+1}$$

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

$(I+\tau A)\vec{u}^{n+1} = \vec{u}^n + \tau \vec{f}^{n+1}$ solve a linear system

$$\|\vec{u}^{n+1}\|_\infty \le \|\vec{u}^n\| + \tau\|\vec{f}^{n+1}\|_\infty$$

$(I+\tau A)\vec{U}^{n+1} = \vec{U}^n + \tau(\vec{f}^{n+1} + \vec{r}^{n+1})$ $\quad \vec{U}^{n+1} = \begin{pmatrix} \vdots \\ u(x_i,t_{n+1}) \\ \vdots \end{pmatrix}$

$\|r^{n+1}\|_\infty \le C(h^2+\tau)$ $\quad C$ indep h,τ

$\vec{e}^{n+1} = \vec{U}^{n+1} - \vec{u}^{n+1}$ $\quad (I+\tau A)\vec{e}^{n+1} = \vec{e}^n + \tau\vec{r}^{n+1}$

$\|\vec{e}^M\| \le CT(h^2+\tau)$

Explicit scheme $\frac{\partial u}{\partial t}(x_i,t_n) - \frac{\partial^2 u}{\partial x^2}(x_i,t_n) = f(x_i,t_n)$

Forward and Finite Diff for $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$

Centered

$$\frac{u(x_i,t_{n+1})-u(x_i,t_n)}{\tau} - \frac{u(x_{i-1},t_n) - 2u(x_i,t_n)+u(x_{i+1},t_n)}{h^2} = f(x_i,t_n) + r_i^n$$

$$|r_i^n| \leq C(h^2+\tau)$$ C depends on $\frac{\partial^4 u}{\partial x^4}$ and $\frac{\partial^2 u}{\partial t^2}$

Scheme (u_i^n are known)

$$\frac{u_i^{n+1}-u_i^n}{\tau} - \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} = f(x_i,t_n)$$

$i = 1,\dots,N$

$u_0^n = 0 \quad u_{N+1}^n = 0$

$\frac{\vec{u}^{n+1}-\vec{u}^n}{\tau} + A\vec{u}^n = \vec{f}^n$

$\vec{u}^{n+1} = (I-\tau A)\vec{u}^n + \tau\vec{f}^n$

(implicit scheme $\vec{u}^{n+1} = (I+\tau A)^{-1}(\vec{u}^n + \tau\vec{f}^{n+1})$) $\quad$ IR: $\frac{1}{1+x} = 1-x+o(x^2)$

Lemma: If $\tau \leq \frac{h^2}{2}$ then

a. $\|\vec{u}^{n+1}\|_\infty \leq \|\vec{u}^n\|_\infty + \tau\|\vec{f}^n\|_\infty$

b. If $\vec{f}^n \geqslant \vec{0}$ and $\vec{u}^n \geqslant \vec{0}$ then $\vec{u}^{n+1} \geqslant \vec{0}$

Proof: $u_i^{n+1} = \left(1-\frac{2\tau}{h^2}\right)u_i^n + \frac{\tau}{h^2}\left(u_{i-1}^n + u_{i+1}^n\right) + \tau f(x_i,t_n)$

$\left(1-\frac{2\tau}{h^2} \geqslant 0\right) \Leftrightarrow \left(\tau \leq \frac{h^2}{2}\right)$ by hypothesis $\quad i = 1,\dots,N$

a. $|u_i^{n+1}| \leq \left|1-\frac{2\tau}{h^2}\right||u_i^n| + \left|\frac{\tau}{h^2}\right|\left(|u_{i-1}^n| + |u_{i+1}^n|\right) + \tau|f(x_i,t_n)|$

Since $1-\frac{2\tau}{h^2} \geq 0$ $\left|1-\frac{2\tau}{h^2}\right| = 1-\frac{2\tau}{h^2}$

$\frac{\tau}{h^2} \geq 0$ $\left|\frac{\tau}{h^2}\right| = \frac{\tau}{h^2}$

$$|u_i^{n+1}| \leq \underbrace{\left(1-\frac{2\tau}{h^2}+\frac{2\tau}{h^2}\right)}_{1} \max_{1\leq j\leq N} |u_j^n| + \tau\left|f(x_i,t_n)\right|$$

$\left(0=|u_0^n| \leq \max_{1\leq j\leq N} |u_j^n|\right)$

$i=1,\dots,N$

Therefore $\max_{1\leq i\leq N} |u_i^{n+1}| \leq \max_{1\leq j\leq N} |u_j^n| + \tau \max_{1\leq i\leq N} f(x_i,t_n)$

Remark: If $\tau \geq \frac{h^2}{2}$, you may have blow up

Thm: convergence of the explicit scheme in the $\|\cdot\|_\infty$

If $u \in \mathcal{C}^4$ in space and $\mathcal{C}^2$ in time, if $\tau \leq \frac{h^2}{2}$

then there exists D indep of h and τ such that $(t_M = T)$

$$\|e^M\|_\infty = \|\vec{U}^M - \vec{u}^M\|_\infty \leq D(h^2+\tau) \qquad \tau = T/M$$

Proof: $\vec{U}^{n+1}$ satisfies $\vec{U}^{n+1} = (I-\tau A)\vec{U}^n + \tau(\vec{\varphi}^n + \vec{r}^n)$

with $\|\vec{r}^n\|_\infty \leq C(h^2+\tau)$

Stability in the $\|\cdot\|_2$ norm

Explicit scheme $\vec{u}^{n+1} = (I - \tau A)\vec{u}^n + \tau \vec{f}^n$

Implicit $\vec{u}^{n+1} = (I + \tau A)^{-1}\left(\vec{u}^n + \tau \vec{f}^n\right)$

A is sym. pos. def $\quad A\vec{\varphi}_i = \lambda_i \vec{\varphi}_i \qquad \vec{\varphi}_i^T \vec{\varphi}_j = \delta_{ij}$

$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N \qquad i,j = 1,\dots,N$

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \qquad \lambda_i = 2\left(1 - \cos\frac{\pi i}{N+1}\right)(N+1)^2$$

$\lambda_1 \sim \pi^2 \qquad \lambda_N = \frac{4}{h^2} + O\left(\frac{1}{h^4}\right)$

Lemma:

a) $B \in \mathbb{R}^{N\times N}$ symmetric $\quad B\vec{\varphi}_i = \mu_i \vec{\varphi}_i \quad i = 1,\dots,N \quad \mu_i \in \mathbb{R} \quad \vec{\varphi}_i^T \vec{\varphi}_j = \delta_{ij}$

$\forall \vec{x} \in \mathbb{R}^N \quad \|B\vec{x}\|_2 \leq \max_{1\leq i\leq N} |\mu_i| \, \|\vec{x}\|_2$

$\left(\|B\|_2 = \sup_{\substack{\vec{x}\in\mathbb{R}^N \\ \vec{x}\neq 0}} \frac{\|B\vec{x}\|_2}{\|\vec{x}\|_2}, \forall \vec{x} \in \mathbb{R}^N \ \|B\vec{x}\|_2 \leq \|B\|_2 \|\vec{x}\|_2\right)$

Proof: $\forall \vec{x}$

B

b. $\|\vec{u}^n$

C_0

b. Implicit scheme $\forall \tau > 0 \quad \|\bar{u}^{n+1}\|_2 \leq \|\bar{u}^n\|_2 + \tau \|\bar{f}^{n+1}\|_2$

c. Explicit —— If $\tau \leq \frac{2}{\lambda_N}$ $\|\bar{u}^{n+1}\|_2 \leq \|\bar{u}^n\| + \tau \|\bar{f}^n\|_2$

Proof: $\forall \bar{x} \in \mathbb{R}^N$, $\bar{x} = \sum_{j=1}^{N} x_j \bar{\psi}_j$ $\quad \|\bar{x}\|^2 = \bar{x}^T \bar{x} = \sum_{j=1}^{N} x_j^2$

$$B\bar{x} = B \sum_{j=1}^{N} x_j \psi_j = \sum_{j=1}^{N} x_j \mu_j \vec{\psi}_j$$

$$\|B\bar{x}\|_2^2 = \sum_{j=1}^{N} x_j^2 \mu_j^2 \leq \max_{1 \leq i \leq N} \mu^2 \sum_{j=1}^{N} x_j^2 = \|\bar{x}\|_2^2$$

Take the square root

b. $\|\bar{u}^{n+1}\|_2 = \|(I+\tau A)^{-1}(\bar{u}^n + \tau \bar{f}^{n+1})\|_2$ eigenvalues of $(I+\tau A)^{-1}$: $\frac{1}{1+\tau\lambda_i} \leq 1$ $i=1,\dots,N$ ($\lambda_i \geq 0$)

$$\overset{a}{\leq} 1 \cdot \|\bar{u}^n + \tau \bar{f}^{n+1}\|_2 \leq \|\bar{u}^n\|_2 + \tau \|\bar{f}^{n+1}\|_2$$

true $\forall \tau > 0$

c. $\|\bar{u}^{n+1}\|_2 \leq \|(I-\tau A)\bar{u}^n\|_2 + \tau \|\bar{f}^n\|_2$

$\leq \max_{1 \leq i \leq N} |1-\tau\lambda_i| \, \|\bar{u}^n\|_2 + \tau \|\bar{f}^n\|_2$

$|1-\tau\lambda_i| \leq 1$ $i=1,\dots,N$ if $-1 \leq 1-\tau\lambda_i \leq 1$

that is to say $\tau \leq \frac{2}{\lambda_i}$ $i=1,\dots,N$

which is the case if $\tau \leq \frac{2}{\lambda_N}$



If $\tau \leq \frac{2}{\lambda_N}$ then $\|\vec{u}^{n+1}\|_2 \leq \|\vec{u}^n\|_2 + \tau \|\vec{f}^n\|_2$

Reminder $\vec{U}^{n+1} = \begin{pmatrix} u(x_1, t_{n+1}) \\ \vdots \\ u(x_N, t_{n+1}) \end{pmatrix}$

Explicit: $\vec{U}^{n+1} = (I - \tau A)\vec{U}^n + \tau(\vec{f}^n + \vec{r}^n)$, $\|\vec{r}^n\|_\infty \leq C(h^2 + \tau)$

Implicit: $\vec{U}^{n+1} = (I + \tau A)^{-1}(\vec{U}^n + \tau(\vec{f}^{n+1} + \vec{r}^{n+1}))$ $\|\vec{r}^{n+1}\|_\infty \leq C(h^2 + \tau)$

Thm: convergence in the $\|\cdot\|_2$ norm

Implicit: $\forall \tau > 0$ $\|\vec{e}^{n+1}\|_2 \leq \|\vec{e}^n\|_2 + \tau \|\vec{r}^{n+1}\|_2$

$\|\vec{e}^M\|_2 \leq CT \frac{h^2 + \tau}{h^{1/2}}$

Explicit: If $\tau \leq \frac{2}{\lambda_N}$ $\|e^{n+1}\|_2 \leq \|\vec{e}^n\|_2 + \tau \|\vec{r}^n\|_2$

$\|\vec{r}^n\|_2^2 = \sum_{i=1}^N (r_i^n)^2 \leq N \|\vec{r}^n\|_\infty^2$ $\|\vec{e}^M\|_2 \leq CT \frac{h^2 + \tau}{h^{1/2}}$

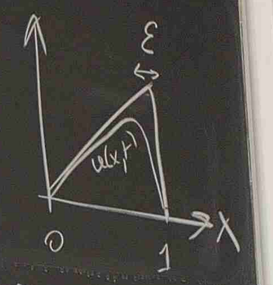
$h = \frac{1}{N+1}$ $N \leq N+1 = \frac{1}{h}$

Extensions :

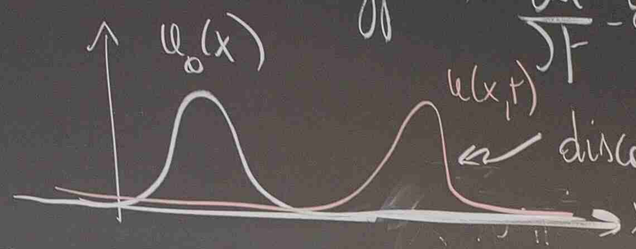
- advection diffusion $\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = 1$

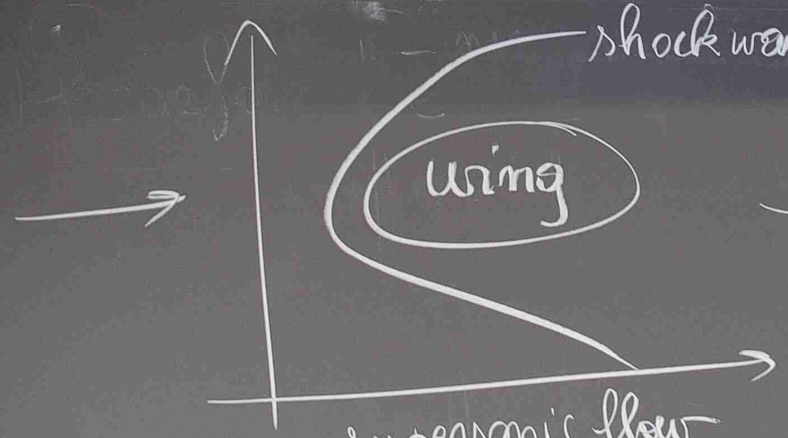
$\varepsilon > 0$

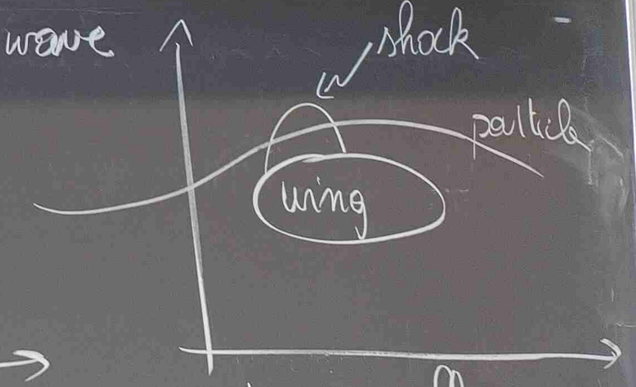
$0 < \varepsilon << 1$



- Nonlinear advection-diffusion $\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} = 0$







Toy problem : Navier-Stokes $\rho\left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u\right) - \mu \Delta u + \nabla p = 0$

$\operatorname{div} u = 0$

Higher order methods in time:

space discretization only

$$\frac{\partial u}{\partial t}(x_i,t) - \frac{\partial^2 u}{\partial x^2}(x_i,t) = f(x_i,t)$$

$$\frac{\partial u}{\partial t}(x_i,t) - \frac{u(x_{i-1},t) - 2u(x_i,t) + u(x_{i+1},t)}{h^2} = f(x_i,t) + r_i(t)$$

$u_i(t)$ approx of $u(x_i,t)$

$$u_i'(t) - \frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{h^2} = f(x_i,t) \qquad i = 1, \dots, N$$

ODE system

$$\vec{u}'(t) + A\vec{u}(t) = \vec{f}(t)$$

$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix} \qquad A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

Implicit, order 2 scheme $\vec{u}^n$ approx of $\vec{u}(t_n)$

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} + A\,\frac{\vec{u}^n + \vec{u}^{n+1}}{2} = f(t_{n+1/2})$$

Explicit RK2

$$\begin{aligned} y'(t) &= f(t, y(t)) \\ k_1 &= f(t_0, y_0) \\ k_2 &= f(t_0 + \tau/2,\ y_0 + \frac{\tau}{2}k_1) \\ y_1 &= y_0 + \tau k_2 \end{aligned}$$

$$\vec{k}_1 = f(t_n) - A\vec{u}^n$$

$$\vec{k}_2 = f(t_n + \tau/2) - A(\vec{u}^n + \frac{\tau}{2}\vec{k}_1)$$

$$\vec{u}^{n+1} = \vec{u}^n + \tau\vec{k}_2$$

Extend to dimension 2 and 3 in space

$\frac{\partial^2 u}{\partial x^2}$ becomes $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ dimension 2

$\vec{u}'(t) + A\vec{u}(t) = \vec{f}(t)$ $\quad A = \frac{1}{h^2}\left(I_2 \otimes I_2 + I_2 \otimes I_2\right)$ $\quad A_1$

2.1. Elliptic $-\Delta u = f \Omega$
$u = g \partial\Omega$

2.2. Parabolic $\frac{\partial u}{\partial t} - \Delta u = f$

2.3. Hyperbolic
- transport (convection advection)
- wave (propagation)



transport:

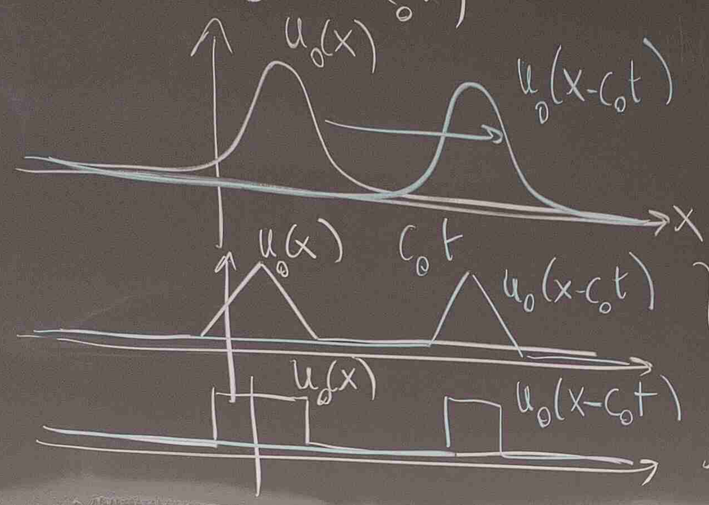
Given $u_0(x)$ $x \in \mathbb{R}$ initial cond ($0 \le x \le 1$ later)
$c(x,t)$ velocity $x \in \mathbb{R}, t > 0$
$f(x,t)$ source

Find $u(x,t)$ such that

$$(1) \begin{cases} \frac{\partial u}{\partial t}(x,t) + c(x,t)\frac{\partial u}{\partial x}(x,t) = f(x,t) & x \in \mathbb{R}\ t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

- Solution when $f(x,t)=0$ $\quad c(x,t)=c_0=cst$

$$u(x,t)=u_0(x-c_0t)$$



weak solutions of the transp. eq.

- when $c(x,t)$ is not constant:

introduce $X'(t)=c(X(t),t)$ (2) (c is $\mathcal{C}^1$) trajectories

$$\frac{d}{dt}u(X(t),t)=\frac{\partial u}{\partial t}(X(t),t)+\frac{\partial u}{\partial x}(X(t),t)\underbrace{X'(t)}_{=c(X(t),t)}$$

Since u satisfies the transp. eq

$$\frac{d}{dt}u(X(t),t)=f(X(t),t)$$

integrate between $t=0$ and $t=T$:

$$u(X(T),T)=u(X(0),0)+\int_0^T f(X(s),s)\,ds$$

where $X(t)$ solves (2)

For instance if $c(x,t)=x$ and $f=0$ then

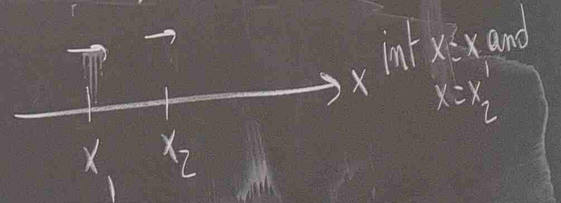
$$X(t)=X(0)\,e^t$$

$$u(\underbrace{X(0)e^T}_{x},T)=u(X(0),0)$$

$$u(x,T)=u(xe^{-T},0)$$

- conservative form:

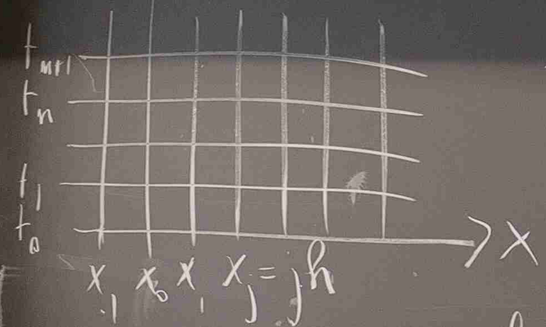
$$\frac{\partial u}{\partial t}+\underbrace{\frac{\partial}{\partial x}(cu)}_{c\frac{\partial u}{\partial x}+\frac{\partial c}{\partial x}u}=f$$



$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx + \Big[c(x,t)u(x,t)\Big]_{x=x_1}^{x=x_2} = \int_{x_1}^{x_2} f(x,t)\,dx$$

rate of change — amount of pollutant in (x_1,x_2) — flux of pollutant entering or leaving (x_1,x_2) — source or sink of pollutant

Numerical schemes ?



$x_j = jh$ h space step $j \in \mathbb{Z}$

$t_n = n\tau$ τ time step $n \in \mathbb{N}$

approximate $u(x_j,t_n)$ by u_j^n

Centered scheme, explicit: $\frac{u_j^{n+1}-u_j^n}{\tau} + c(x_j,t_n)\frac{u_{j+1}^n - u_{j-1}^n}{2h} = f(x_j,t_n)$

$\left(\frac{\partial u}{\partial t}(x_j,t_n) + c(x_j,t_n)\frac{\partial u}{\partial x}(x_j,t_n) = f(x_j,t_n)\right)$

Forward FDT — centered FDT

The centered scheme is always unstable!

$f=0 \quad c(x,t)=c_0=cst \quad \frac{u_j^{n+1}-u_j^n}{\tau}+c_0\frac{u_{j+1}^n-u_{j-1}^n}{2h}=0 \quad j\in\mathbb{Z}$ mis fixed

$n=0 \quad u_j^0=u_0(x_j)$

$n=1 \quad u_j^1$ with the scheme

time marching

$u(x,t)=u_0(x-c_0t)$

Case where $u_0(x)=e^{imx}$ $m\in\mathbb{N}$ (think of Fourier series)

$|u(x,t)|=|e^{im(x-c_0t)}|=1$

$u_j^1=u_j^0-\frac{c_0\tau}{2h}(u_{j+1}^0-u_{j-1}^0)$

$=e^{imjh}-\frac{c_0\tau}{2h}(e^{im(j+1)h}-e^{im(j-1)h})$

$u_j^1=e^{imjh}\left(1-\frac{c_0\tau}{2h}2i\sin(mh)\right)$

by induction:

$$u_j^n = e^{imjh}\left(1 - \frac{c_0\tau}{h} i\sin(mh)\right)^n$$

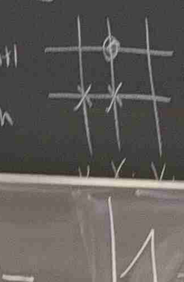
$$|u_j^n| = \underbrace{|e^{imjh}|}_{1}\left(\underbrace{\sqrt{1+\left(\frac{c_0\tau}{h}\right)^2 \sin^2(mh)}}_{\geq 1 \text{ unless } mh = k\pi}\right)^n$$

$|u_j^n| \xrightarrow{n\to\infty} +\infty$!

Upwind scheme: $f=0$ $c(x,t) = c_0 = \text{cst} > 0$

$$\frac{u_j^{n+1} - u_j^n}{\tau} + c_0 \frac{u_j^n - u_{j-1}^n}{h} = 0 \qquad \forall n \text{ fixed}$$

$c_0 > 0$ →



$u_0(x) = e^{imx}$

$$u_j^1 = u_j^0 - \frac{c_0\tau}{h}\left(u_j^n - u_{j-1}^n\right)$$

$$= e^{imjh} - \frac{c_0\tau}{h}\left(e^{imjh} - e^{im(j-1)h}\right)$$

$$u_j^1 = e^{imjh}\left(1 - \frac{c_0\tau}{h}\left(1 - e^{-imh}\right)\right)$$

$u_j^n = \cdots$

$$|u_j^n| = \underbrace{\left|1 - \frac{c_0\tau}{h}\left(1 - e^{-imh}\right)\right|^n}_{\leq 1 \text{ if } \frac{c_0\tau}{h} \leq 1}$$

$|u_j^n|$ remains bounded when $n\to\infty$ if $\frac{c_0\tau}{h} \leq 1$

This is called L^2 stability $(L^2(\mathbb{R}))$

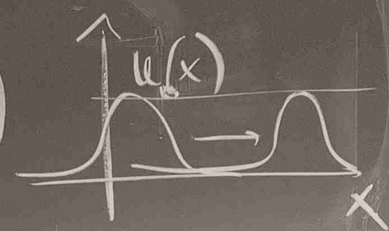
Let us study L^∞ stability

Lemma: $f=0$ $\quad c(x,t)=c_0\geq 0$ $\quad \frac{u_j^{n+1}-u_j^n}{\tau}+c_0\frac{u_j^n-u_{j-1}^n}{h}=0$ $\quad j\in\mathbb{Z}$

If $\tau\leq\frac{h}{c_0}$ then

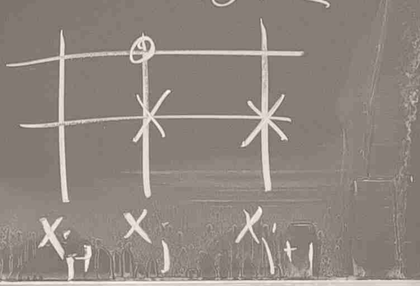
a. If $(u_j^n\geq 0 \quad j\in\mathbb{Z})$ then $(u_j^{n+1}\geq 0 \quad j\in\mathbb{Z})$

b. $\sup_{j\in\mathbb{Z}}|u_j^{n+1}|\leq\sup_{j\in\mathbb{Z}}|u_j^n|$



Remark: if $c_0<0$ $\quad \frac{u_j^{n+1}-u_j^n}{\tau}+c_0\frac{u_{j+1}^n-u_j^n}{h}=0$

stable if $\tau\leq\frac{h}{|c_0|}$



Proof: $u_j^{n+1}=\left(1-\frac{c_0\tau}{h}\right)u_j^n+\frac{c_0\tau}{h}u_{j-1}^n$

a. $1-\frac{c_0\tau}{h}\geq 0\iff\tau\leq\frac{h}{c_0}$ then if $u_j^n\geq 0$ $j\in\mathbb{Z}$

$u_j^{n+1}\geq 0$ —

b. $|u_j^{n+1}|=\left|\left(1-\frac{c_0\tau}{h}\right)u_j^n+\frac{c_0\tau}{h}u_{j-1}^n\right| \quad \forall j\in\mathbb{Z}$

$\leq\underbrace{\left|1-\frac{c_0\tau}{h}\right|}_{=1-\frac{c_0\tau}{h}}|u_j^n|+\underbrace{\left|\frac{c_0\tau}{h}\right|}_{\frac{c_0\tau}{h}}|u_{j-1}^n|\leq\underbrace{\left(1-\frac{c_0\tau}{h}+\frac{c_0\tau}{h}\right)}_{1}\sup_{h\in\mathbb{Z}}|u_h^n|$

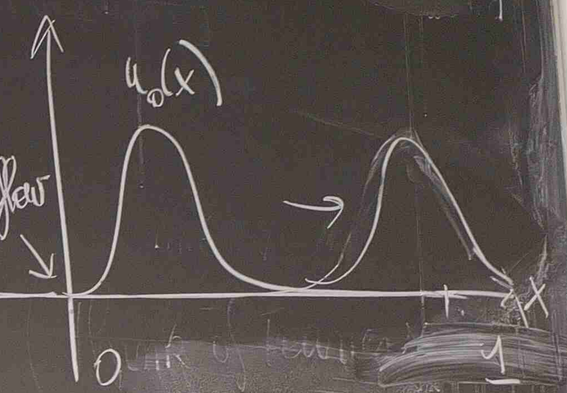
What happens if $0 \le x \le 1$?

$f(x,t) = 0$ $\quad c(x,t) = c_0 \ge 0$

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + c_0 \dfrac{\partial u}{\partial x}(x,t) = 0 & 0<x<1 \quad t>0 \\ u(x,0) = u_0(x) \\ u(0,t) = 0 \ \text{(for instance)} \end{cases}$$

$u(x,t) = u_0(x - c_0 t)$

where u_0 is extended by 0 for $x \le 0$



scheme: $c_0 > 0$

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\tau} + c_0 \dfrac{u_j^n - u_{j-1}^n}{h} = 0 & j = 1, \dots, N+1 \\ u_0^n = 0 \end{cases}$$

$x=0$ $\qquad$ $x=1$

x_0 x_1 x_j x_{N+1}

$x_j = jh \quad h = \dfrac{1}{N+1}$

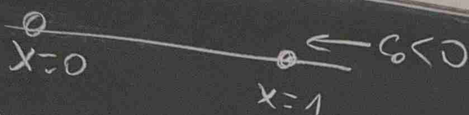
$j = 0, 1, \dots, N, N+1$

Lemma: if $\tau \le \dfrac{h}{c_0}$ then

a. If $\left(u_j^n \ge 0 \quad j = 1, \dots, N+1 \right)$ then $\left(u_j^{n+1} \ge 0 \quad j = 1, \dots, N+1 \right)$

b. $\max_{1 \le j \le N+1} |u_j^{n+1}| \le \max_{1 \le j \le N} |u_j^n|$

Remarks:



- If $c_0<0$

$$\begin{cases} \frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial x}(x,t)=0 & 0<x<1\ t>0 \\ u(x,0)=u_0(x) \\ u(1,t)=0 \end{cases}$$

scheme:

$$\begin{cases} \frac{u_j^{n+1}-u_j^n}{\tau}+c_0\frac{u_{j+1}^n-u_j^n}{h}=0 & j=0,\dots,N \\ u_{N+1}^n=0 \end{cases}$$

stable if $\tau\le\frac{h}{|c_0|}$

- the upwind scheme ($c_0>0$ or $c_0<0$) is stable if $\tau\le\frac{h}{|c_0|}$ and $O(h+\tau)$ convergent (backward or foward FDF)

Extensions:

- $\frac{\partial u}{\partial t}(x,t)+c(x,t)\frac{\partial u}{\partial x}(x,t)=f(x,t)$ $\quad 0<x<1\ t>0$

$u(x,0)=u_0(x)$ $\quad 0<x<1$

if $c(0,t)>0$ then $u(0,t)=0$ (or something)

if $c(1,t)<0$ $\quad u(1,t)=0$



scheme: if $c(x_j,t_n)\ge 0$ then

$$\frac{u_j^{n+1}-u_j^n}{\tau}+c(x_j,t_n)\frac{u_j^n-u_{j-1}^n}{h}=f(x_j,t_n)$$

if $c(x_j,t_n)\le 0$ then

$$\frac{u_j^{n+1}-u_j^n}{\tau}+c(x_j,t_n)\frac{u_{j+1}^n-u_j^n}{h}=f(x_j,t_n)$$

1

- Conservative transport eq. $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(cu) = f$

same upwind scheme: if $c(x_j, t_n) \geq 0$

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{c(x_j, t_n) u_j^n - c(x_{j-1}, t_n) u_{j-1}^n}{h} = f(x_j, t_n)$$

- High order scheme? $O(h^2 + \tau^2)$

Lax-Wendroff $\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} = 0$

$$u(x, t_{n+1}) = u(x, t_n) + \tau \frac{\partial u}{\partial t}(x, t_n) + \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2}(x, t_n) + O(\tau^3)$$

$$\frac{\partial u}{\partial t}(x_j, t_n) + c_0 \frac{\partial u}{\partial x}(x_j, t_n) = 0$$

$$\frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\tau} - \frac{\tau}{2} \frac{\partial^2 u}{\partial t^2}(x_j, t_n) + c_0 \frac{\partial u}{\partial x}(x_j, t_n) = O(\tau^2)$$

$c_0^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_n)$

use only centered finite diff. formula $O(h^2)$

scheme : $\dfrac{u_j^{n+1}-u_j^n}{\tau} - \dfrac{c_0^2\tau}{2}\dfrac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} + c_0\dfrac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$

stable if $\tau \leq \dfrac{h}{|c_0|}$

Remark: upwind scheme = centered scheme + diffusion (numerical)

$\dfrac{u_j^{n+1}-u_j^n}{\tau} + c_0\dfrac{u_j^n - u_{j-1}^n}{h} = 0 \qquad (c_0 > 0 \ \ f = 0)$

$\dfrac{u_j^{n+1}-u_j^n}{\tau} + c_0\dfrac{u_{j+1}^n - u_{j-1}^n}{2h} = \dfrac{ch}{2}\dfrac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} = 0$

2.3 Hyperbolic

transport wave

↳ in 2D

Recall $\frac{\partial u}{\partial t}(x,t) + c(x,t)\frac{\partial u}{\partial x}(x,t) = f(x,t)$

$0 < x < 1 \quad t > 0$

$u(x,0) = u_0(x)$

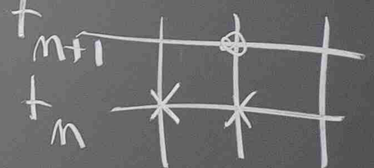
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if $c(0,t) > 0$ then $u(0,t) = 0$

$c(1,t) < 0$ then $u(1,t) = 0$

upwind scheme

if $c(x_j, t_n) > 0$ $\longrightarrow$



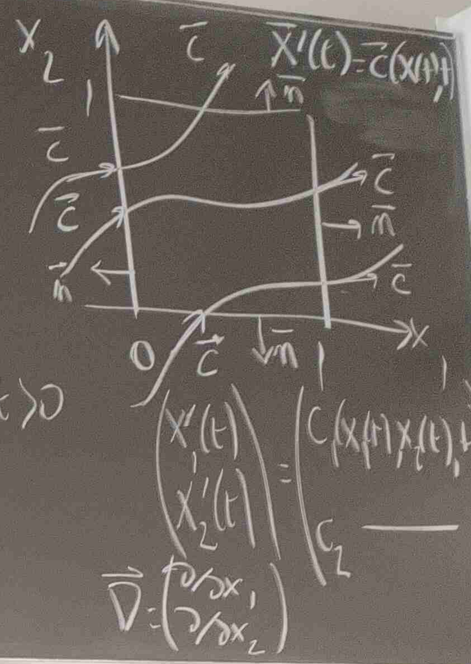
$$\frac{u_j^{n+1} - u_j^n}{\tau} + c(x_j, t_n)\frac{u_j^n - u_{j-1}^n}{h} = f(x_j, t_n)$$

2D?

$$\begin{cases} \dfrac{\partial u}{\partial t}(x_1,x_2,t) + \vec{c}(x_1,x_2,t)\cdot\vec{\nabla}u(x_1,x_2,t) = f(x_1,x_2,t) \\ \qquad\qquad (x_1,x_2)\in(0,1)^2 \quad t>0 \\ u(x_1,x_2,0) = u_0(x_1,x_2) \\ u(x_1,x_2,t) = 0 \end{cases}$$

on $\Gamma_-^{\vec{c}} = \{(x_1,x_2)\in\partial(0,1)^2;\ \vec{c}\cdot\vec{n}<0\}, t>0$

inflow

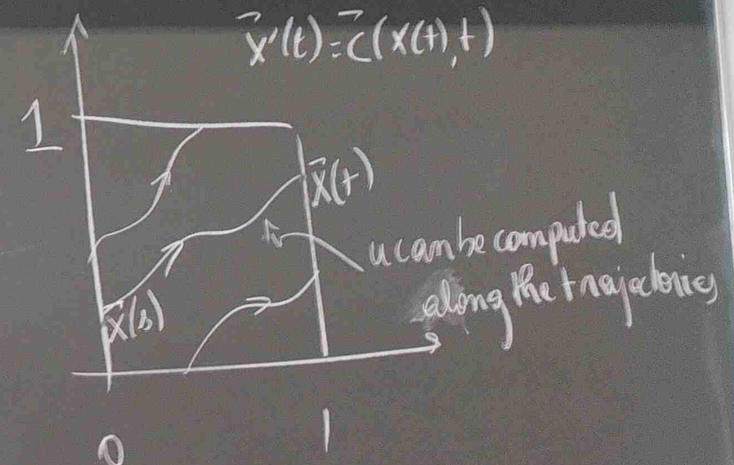


Why? 2 reasons:

0. Let $\vec{X}(t)$ be the trajectories def by $\vec{X}'(t) = \vec{c}(\vec{X}(t),t)$ then

$$\frac{d}{dt}u(\vec{X}(t),t) = f(\vec{X}(t),t)$$

so that $u(\vec{X}(t),t) - u(\vec{X}(s),s) = \int_s^t f(\vec{X}(\alpha),\alpha)\,d\alpha$



◦ Another reason: a priori L^2 estimate of u:

mult. the transp. eq. by $u(x_1, x_2, t)$ and $\iint_0^1 dx_1 dx_2$

$$\int_\Omega \frac{\partial u}{\partial t} u \, dx_1 dx_2 + \int_\Omega (\vec{c} \cdot \vec{\nabla} u) u \, dx_1 dx_2 = \iint_\Omega u \, dx_1 dx_2 \qquad \Omega = (0,1)^2$$

Special case where $f = 0$ $\operatorname{div} \vec{c} = 0$

$$\frac{1}{2} \frac{d}{dt} \int_\Omega (u(x_1, x_2, t))^2 dx_1 dx_2 + \int_\Omega (\vec{c} \cdot \vec{\nabla} u) u \, dx_1 dx_2 = 0$$

$$\operatorname{div}\left(\vec{c} \frac{u^2}{2}\right) = \frac{u^2}{2} \underbrace{\operatorname{div} \vec{c}}_{0} + \vec{c} \cdot \nabla\left(\frac{u^2}{2}\right) = (\vec{c} \cdot \vec{\nabla} u) u$$

$$\int_\Omega (\vec{c} \cdot \vec{\nabla} u) u \, dx_1 dx_2 = \int_\Omega \operatorname{div}\left(\vec{c} \frac{u^2}{2}\right) dx_1 dx_2 = \int_{\partial\Omega} \vec{c} \cdot \vec{n} \, \frac{u^2}{2} \, ds$$

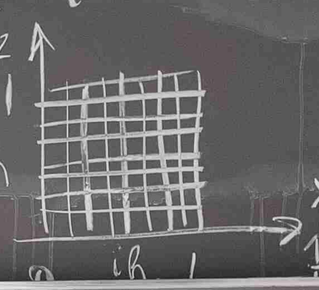
$$= \int_{\Gamma_-} \underbrace{\vec{c} \cdot \vec{n}}_{<0} \underbrace{\frac{u^2}{2}}_{\neq 0} ds + \int_{\partial\Omega \setminus \Gamma_-} \underbrace{\vec{c} \cdot \vec{n}}_{\geq 0} \frac{u^2}{2} ds \geqslant 0$$

Finally $\frac{1}{2}\frac{d}{dt}\int_\Omega (u(x_1,x_2,t))^2 dx_1 dx_2 \le 0$

$$\int_\Omega (u(x_1,x_2,T))^2 dx_1 dx_2 \le \int_\Omega (u(x_1,x_2,0))^2 dx_1 dx_2$$

Upwind scheme?

$u(ih, jh, t_n)$ approx u_{ij}^n

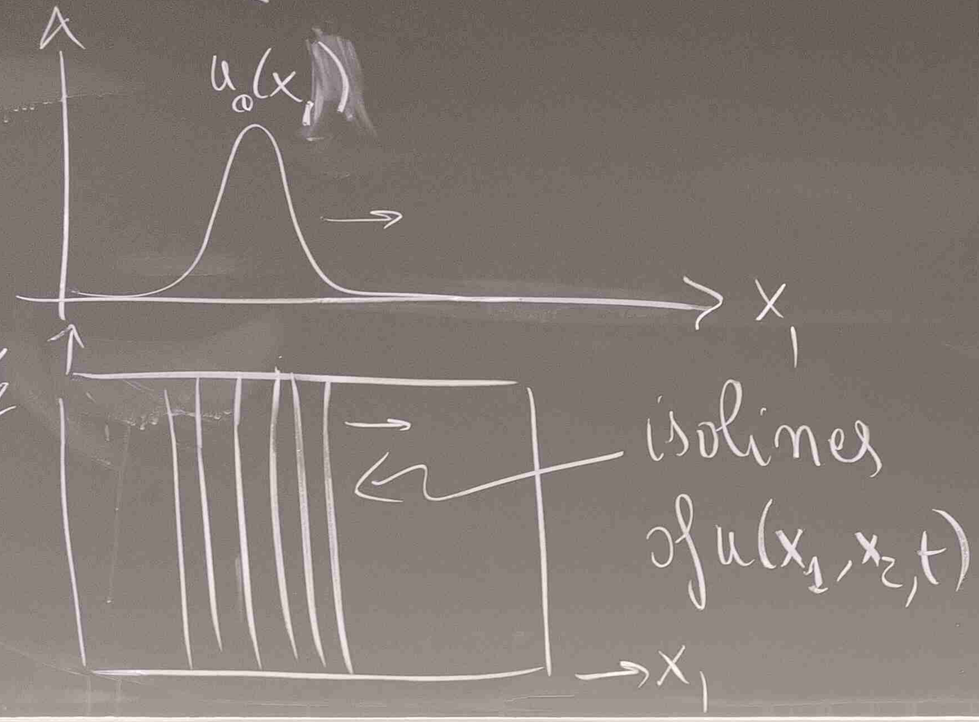


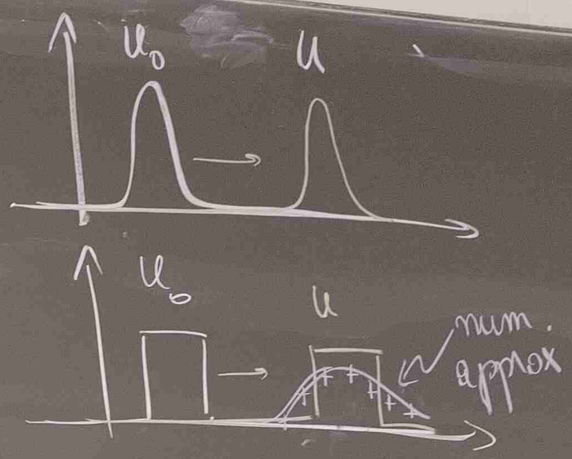
$h = \frac{1}{L+1}$

If $c_1(ih, jh, t_n) > 0$ and $c_2(ih, jh, t_n) > 0$ then < 0

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\tau} + c_1(ih, jh, t_n)\frac{u_{ij}^n - u_{i-1,j}^n}{h} + c_2(ih, jh, t_n)\frac{u_{ij}^n - u_{i,j-1}^n}{h} = f(ih, jh, t_n)$$

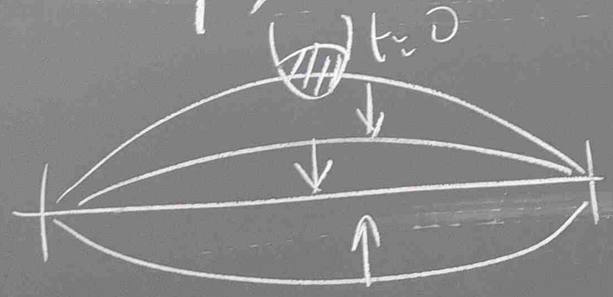
Test case





Wave equation

$c > 0$ propagation speed
$f(x,t)$ source term, $u_0(x), v_0(x)$
$u(x,t)$ unknown



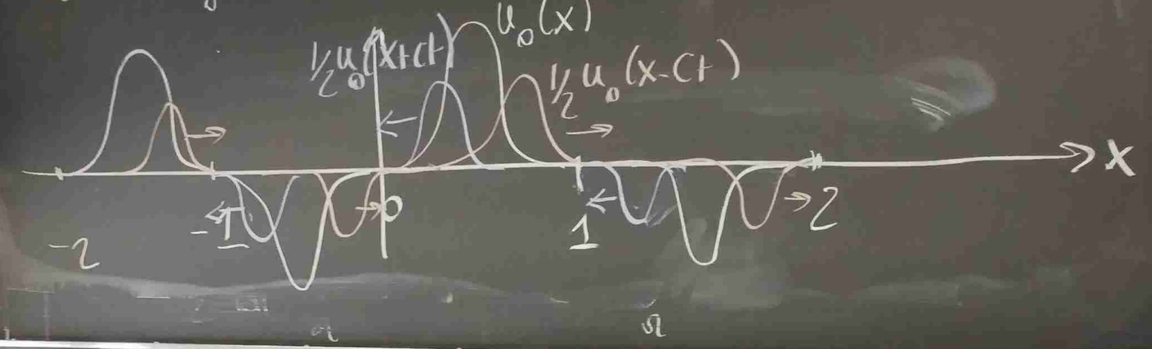
vibrating string

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) & 0<x<1 \quad t>0 \\ u(0,t) = 0 \quad u(1,t) = 0 & t>0 \\ u(x,0) = u_0(x) & 0<x<1 \\ \dfrac{\partial u}{\partial t}(x,0) = v_0(x) \end{cases}$$

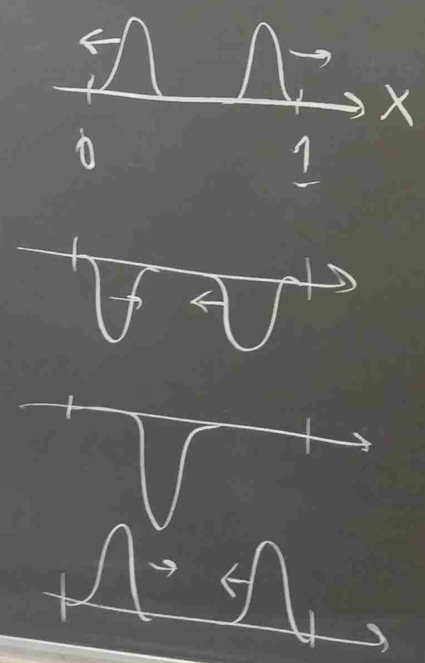
Remarks:

- $f=0$ $v_0=0$ then $u(x,t)=\frac{1}{2}\left(u_0(x+ct)+u_0(x-ct)\right)$

where u_0 is a 2-periodic, odd extension of u_0



D'Alembert principle

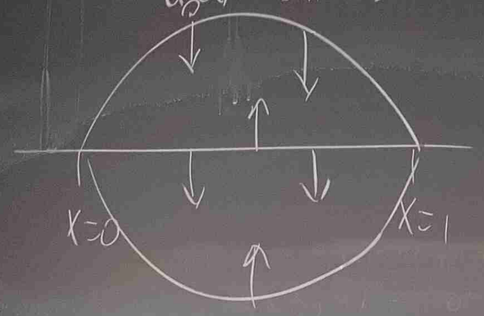


- $f=0$ $v_0=0$ $u_0(x)=\sin(\pi x)$ then

$$u(x,t)=\sin(\pi x)\cos(\pi c t)$$

first term in the Fourier expansion

$u_0(x)=\sin(\pi x)$



- A priori energy estimate (cons. of energy)

mult. the wave eq by $\frac{\partial u}{\partial t}(x,t)$ (if u: deformation, $\frac{\partial u}{\partial t}$: velocity)

and $\int_0^1 dx$

$$\int_0^1 \frac{\partial^2 u}{\partial t^2}(x,t)\frac{\partial u}{\partial t}(x,t)\,dx - c^2\int_0^1 \frac{\partial^2 u}{\partial x^2}(x,t)\frac{\partial u}{\partial t}(x,t)\,dx = \underbrace{\int_0^1 f(x,t)\frac{\partial u}{\partial t}(x,t)\,dx}_{\text{power}}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\partial u}{\partial t}(x,t)\right)^2 dx + c^2 \int_0^1 \frac{\partial u}{\partial x}(x,t)\frac{\partial}{\partial x}\frac{\partial u}{\partial t}(x,t)\,dx = \left[\frac{\partial u}{\partial x}(x,t)\frac{\partial u}{\partial t}(x,t)\right]_{x=0}^{x=1}$$

kinetic en.

$$= \int_0^1 f(x,t)\frac{\partial u}{\partial t}(x,t)\,dx$$

$$(uv)' = u'v + uv'$$
$$[uv] = [u'v + uv']$$
$$u \to u'$$
$$[u'v'] = [u''v + u'v']$$

$$\frac{\partial u}{\partial t}(0,t) = \lim_{\varepsilon \to 0} \frac{u(\varepsilon,t) - u(0,t)}{\varepsilon}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + c^2\left(\frac{\partial u}{\partial x}(x,t)\right)^2\right)dx = \int_0^1 f(x,t)\frac{\partial u}{\partial t}(x,t)\,dx$$

time deriv

kinetic en. + deform. en.

= power forces

$f = 0$:

$$\int_0^1 \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + c^2\left(\frac{\partial u}{\partial x}(x,t)\right)^2\right)dx = cst$$

$f \neq 0$:

$$\frac{1}{2}\frac{d}{dt}y(t) \leq \underbrace{\left(\int_0^1 f(x,t)^2\,dx\right)^{1/2}}_{C(t)} \underbrace{\left(\int_0^1 \left(\frac{\partial u}{\partial t}(x,t)\right)^2 dx\right)^{1/2}}_{(y(t))^{1/2}}$$

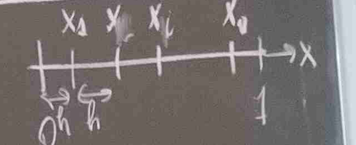
$$y(t) = \left(\int_0^1 \left(\frac{\partial u}{\partial t}(x,t)\right)^2 dx\right)^{1/2}$$

$$\frac{1}{2}\frac{y'(t)}{(y(t))^{1/2}} \leq C(t)$$

$$\frac{d}{dt}(y(t))^{1/2} \leq C(t)$$

$$(y(T))^{1/2} - (y(0))^{1/2} \leq \int_0^T C(t)\,dt$$

Space discretisation



$$h = \frac{1}{N+1} \quad x_i = ih \quad i = 0, \ldots, N+1$$

compute $u_i(t)$ approx $u(x_i, t)$ $i = 1, \ldots, N$

Write the wave eq at x_i

$$\frac{\partial^2 u}{\partial t^2}(x_i,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x_i,t) = f(x_i,t)$$

$$\frac{\partial^2 u}{\partial t^2}(x_i,t) - c^2 \frac{2u(x_{i-1},t) - 2u(x_i,t) + u(x_{i+1},t)}{h^2} + O(h^2) = f(x_i,t)$$

Replace $u(x_i,t)$ by $u_i(t)$, forget $O(h^2)$ if u C^4 space

$$u_i''(t) - c^2 \frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{h^2} = f(x_i,t) \qquad i = 1,\dots,N$$

with $u_0(t) = 0$ and $u_{N+1}(t) = 0$

Differential system

$$\vec{u}''(t) + A\,\vec{u}(t) = \vec{f}(t)$$

$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix} \qquad A = \frac{c^2}{h^2}\begin{pmatrix} 2 & -1 & & (0) \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ (0) & & -1 & 2 \end{pmatrix} \qquad \vec{f}(t) = \begin{pmatrix} f(x_1,t) \\ \vdots \\ f(x_N,t) \end{pmatrix}$$

$t_n = n\tau$ $n = 0, 1, 2, \dots$

$\vec{u}(t_n)$ approximated by $\vec{u}^n = \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix}$ approx of $\begin{pmatrix} u(x_1, t_n) \\ \\ u(x_N, t_n) \end{pmatrix}$

- Explicit scheme:

$$\frac{\vec{u}^{n-1} - 2\vec{u}^n + \vec{u}^{n+1}}{\tau^2} + A\vec{u}^n = \vec{f}(t_n)$$

if $\vec{u}^{n-1}$ and $\vec{u}^n$ are known, can be used to compute $\vec{u}^{n+1}$

$$\vec{u}^{n+1} = (2I - \tau^2 A)\vec{u}^n - \vec{u}^{n-1} + \tau^2 \vec{f}(t_n)$$

$$\vec{u}^0 = \begin{pmatrix} u_0(x_1) \\ \vdots \\ u_0(x_N) \end{pmatrix} \quad \vec{u}^1 ?$$

stable under condition $\left(\tau \leq \frac{h}{c}\right)$

- Implicit scheme

$$\frac{\vec{u}^{n-1} - 2\vec{u}^n + \vec{u}^{n+1}}{\tau^2} + A\,\frac{\vec{u}^{n-1} + 2\vec{u}^n + \vec{u}^{n+1}}{4} = \frac{\vec{f}(t_{n-1}) + 2\vec{f}(t_n) + \vec{f}(t_{n+1})}{4}$$

stable $\forall h, \tau > 0$

Origin:

- write (1) as a system of first order

$$\begin{cases} \vec{v}' + A\vec{u} = \vec{f} \\ \vec{u}' = \vec{v} \end{cases}$$

- Use Crank-Nicolson scheme ($O(\tau^2)$ implicit)

$$\begin{cases} \dfrac{\vec{v}^{n+1} - \vec{v}^{n}}{\tau} = \dfrac{\vec{f}^{n} + \vec{f}^{n+1}}{2} - A\,\dfrac{\vec{u}^{n} + \vec{u}^{n+1}}{2} \\ \dfrac{\vec{u}^{n+1} - \vec{u}^{n}}{\tau} = \dfrac{\vec{v}^{n} + \vec{v}^{n+1}}{2} \end{cases}$$

- Eliminate $\vec{v}^{n}$ and $\vec{v}^{n+1}$

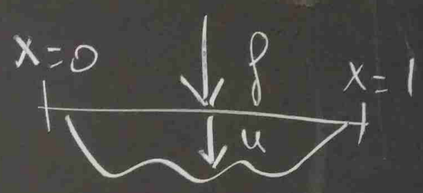
$$\frac{v^{n+1} - v^{n}}{\tau} + \frac{v^{n} - v^{n-1}}{\tau} = \dots$$

$$\frac{v^{n+1} + v^{n}}{\tau} - \frac{v^{n} + v^{n-1}}{\tau} =$$

$$\frac{u^{n+1} - u^{n}}{\tau^2} - \frac{u^{n} - u^{n-1}}{\tau^2} =$$

- The wave eq
- A real life eq: Navier-Stokes eq
- Recap

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f$$



BC $x=0 \quad x=1$

IC $t=0 \quad u_0, v_0$

Expl:

$$\frac{u_i^{n-1} - 2u_i^n + u_i^{n+1}}{\tau^2} - c^2 \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} = f(x_i, t_n)$$

$$\vec{u}^n = \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix}$$

$$\frac{\vec{u}^{n-1} - 2\vec{u}^n + \vec{u}^{n+1}}{\tau^2} + A\vec{u}^n = \vec{f}^n \qquad (1)$$

Impl:

$$\frac{\vec{u}^{n-1} - 2\vec{u}^n + \vec{u}^{n+1}}{\tau^2} + A\left(\frac{\vec{u}^{n-1} + 2\vec{u}^n + \vec{u}^{n+1}}{4}\right) = \vec{f}^n$$

$$A = \frac{c^2}{h^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \qquad (2)$$

or $\vec{f}^{n-1} + 2\vec{f}^n + \vec{f}^{n+1}$ / 4

$v(t)$

$v(t_{n+1}) = v(t_n) + \tau v'(t_n) + \frac{\tau^2}{2} v''(t_n) + \dots$

$v(t_{n-1}) = v(t_n) - \tau v'(t_n) + \frac{\tau^2}{2} v''(t_n) + \dots$

$v(t_{n+1}) + v(t_{n-1}) = 2v(t_n) + \tau^2 v''(t_n) + \dots$

$\frac{v(t_{n-1}) + 2v(t_n) + v(t_{n+1})}{4} = v(t_n) + O(\tau^2)$

Stability: $f = 0$

reproduce $\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + c^2 \left(\frac{\partial u}{\partial x}(x,t)\right)^2 \right) dx$

Impl: take the scalar product with $\vec{u}^{n+1} - \vec{u}^{n-1}$

$\frac{\vec{u}^{n-1} - 2\vec{u}^n + \vec{u}^{n+1}}{\tau^2} + A\frac{\vec{u}^{n-1} + 2\vec{u}^n + \vec{u}^{n+1}}{4} = 0 \quad \left(\vec{u}^{n+1} - \vec{u}^{n-1}\right)$

$\frac{\left(\vec{u}^{n+1} - \vec{u}^n\right) - \left(\vec{u}^n - \vec{u}^{n-1}\right)}{\tau^2}^T \left(\vec{u}^{n+1} - \vec{u}^n + \left(\vec{u}^n - \vec{u}^{n-1}\right)\right) + \left(\vec{u}^n + \vec{u}^{n+1} - \left(\vec{u}^{n-1} + \vec{u}^n\right)\right)^T A \frac{\vec{u}^n + \vec{u}^{n+1} + \vec{u}^{n-1} + \vec{u}^n}{4} = 0$

type $(\vec{a} - \vec{b})^T (\vec{a} + \vec{b}) = \|\vec{a}\|^2 - \|\vec{b}\|^2$

$v^T A v = \langle v, v \rangle_A \quad \langle \vec{a} - \vec{b}, \vec{a} + \vec{b} \rangle_A = \langle \vec{a}, \vec{a} \rangle_A^2 - \langle \vec{b}, \vec{b} \rangle_A^2$

$\left\| \frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} \right\|^2 - \left\| \frac{\vec{u}^n - \vec{u}^{n-1}}{\tau} \right\|^2 + \left(\frac{\vec{u}^n + \vec{u}^{n+1}}{2}\right)^T A \left(\frac{\vec{u}^n + \vec{u}^{n+1}}{2}\right) - \left(\frac{\vec{u}^{n-1} + \vec{u}^n}{2}\right)^T A \left(\frac{\vec{u}^{n-1} + \vec{u}^n}{2}\right) = 0 \quad (3)$

$\vec{a}^T A \vec{a}$

Sum $n = 1$ to $n = M-1$;

Discrete energy conserv.: $\left\| \frac{\vec{u}^M - \vec{u}^{M-1}}{\tau} \right\|^2 + \left(\frac{\vec{u}^{M-1} + \vec{u}^M}{2}\right)^T A \left(\frac{\vec{u}^{M-1} + \vec{u}^M}{2}\right) = \left\| \frac{\vec{u}^1 - \vec{u}^0}{\tau} \right\|^2 + \left(\frac{\vec{u}^0 + \vec{u}^1}{2}\right)^T A \left(\frac{\vec{u}^0 + \vec{u}^1}{2}\right)$

Concl: the implicit scheme is stable $\forall h, \tau > 0$

Exercice: convergence error $\vec{E}^n = \vec{U}^n - \vec{u}^n$ $\vec{U}^n = \begin{pmatrix} u(x_1, t_n) \\ \vdots \\ u(x_N, t_n) \end{pmatrix}$

$$\frac{\vec{E}^{n-1} - 2\vec{E}^n + \vec{E}^{n+1}}{\tau^2} + A\,\frac{\vec{E}^{n-1} + 2\vec{E}^n + \vec{E}^{n+1}}{4} = \vec{f}^n + \vec{r}^n$$

$$|r_i^n| \leq C(h^2 + \tau^2)$$

take the scal. prod with $\vec{E}^{n+1} - \vec{E}^{n-1}$ and conclude...

Expl: Write (1) as (2) + terms:

$$\frac{\vec{u}^{n-1} - 2\vec{u}^n + \vec{u}^{n+1}}{\tau^2} + A\,\frac{\vec{u}^{n-1} + 2\vec{u}^n + \vec{u}^{n+1}}{4} = A\,\frac{\vec{u}^{n-1} - 2\vec{u}^n + \vec{u}^{n+1}}{4}$$

take the scalar product with $\vec{u}^{n+1} - \vec{u}^{n-1}$

We will obtain (3) with an extra term in the rhs

$$\left(\vec{u}^{n+1} - \vec{u}^n + (\vec{u}^n - \vec{u}^{n-1})\right)^T A \left(\frac{\vec{u}^{n+1} - \vec{u}^n - (\vec{u}^n - \vec{u}^{n-1})}{4}\right)$$

$\vec{a} + \vec{b}$ $\qquad$ $\vec{a} - \vec{b}$

The rhs of (3) is now:

$$\text{lhs of (3)} = \left(\frac{\vec{u}^{n+1}-\vec{u}^{n}}{2}\right)^T A\left(\frac{\vec{u}^{n+1}-\vec{u}^{n}}{2}\right) - \left(\frac{\vec{u}^{n}-\vec{u}^{n-1}}{2}\right)^T A\left(\frac{\vec{u}^{n}-\vec{u}^{n-1}}{2}\right) \quad (4)$$

Sum from $n=1$ to $M-1$:

$$\left\|\frac{\vec{u}^{M}-\vec{u}^{M-1}}{2}\right\|^2 + \left(\frac{\vec{u}^{M-1}+\vec{u}^{M}}{2}\right)^T A\left(\frac{\vec{u}^{M-1}+\vec{u}^{M}}{2}\right) + \left(\frac{\vec{u}^{1}-\vec{u}^{0}}{2}\right)^T A\left(\frac{\vec{u}^{1}-\vec{u}^{0}}{2}\right)$$

$$= \left\|\frac{\vec{u}^{1}-\vec{u}^{0}}{2}\right\|^2 + \left(\frac{\vec{u}^{1}+\vec{u}^{0}}{2}\right)^T A\left(\frac{\vec{u}^{1}+\vec{u}^{0}}{2}\right) + \left(\frac{\vec{u}^{M}-\vec{u}^{M-1}}{2}\right)^T A\left(\frac{\vec{u}^{M}-\vec{u}^{M-1}}{2}\right)$$

$$\left\|\frac{\vec{u}^{M}-\vec{u}^{M-1}}{2}\right\|^2 + \left(\frac{\vec{u}^{M-1}+\vec{u}^{M}}{2}\right)^T A\left(\frac{\vec{u}^{M-1}+\vec{u}^{M}}{2}\right)$$

$$\leq \left\|\frac{\vec{u}^{1}-\vec{u}^{0}}{2}\right\|^2 + \left(\frac{\vec{u}^{1}+\vec{u}^{0}}{2}\right)^T A\left(\frac{\vec{u}^{1}+\vec{u}^{0}}{2}\right) + \frac{\lambda_N}{4}\left\|\vec{u}^{M}-\vec{u}^{M-1}\right\|^2$$

$\lambda_1 \|\vec{v}\|^2 \leq \vec{v}^T A \vec{v} \leq \lambda_N \|\vec{v}\|^2$ $\quad A\vec{\varphi}_i = \lambda_i \vec{\varphi}_i$ $\quad \vec{\varphi}_i^T \vec{\varphi}_j = \delta_{ij}$ $\quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$

$$\vec{v} = \sum_{i=1}^{N} \alpha_i \vec{\varphi}_i$$

If $\frac{1}{c^2} - \frac{\lambda_N}{4} \geq 0$ then $\left(\frac{\vec{u}^{M-1}+\vec{u}^{M}}{2}\right)^T A\left(\frac{\vec{u}^{M-1}+\vec{u}^{M}}{2}\right) \leq$ depends only on $\vec{u}^0$ and $\vec{u}^1$

$\zeta^2 \leq \frac{4}{\lambda_N}$ λ_N ?

$A = \frac{c^2}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$

$0 \leq$ eigenv ≤ 4

first Gerschgorin thm

parenthesis

a_{11} a_{ii} a_{NN}

$\lambda \in \bigcup_{i=1}^{N} D_i$

D_i = Ball centered at a_{ii} radius $\sum_{\substack{j=1 \\ j\neq i}}^{N} |a_{ij}|$

$A\underline{\varphi} = \lambda\underline{\varphi}$ $\quad (a_{ii} - \lambda_i)\varphi_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij}\varphi_j$

$A = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}_N$

A sym $\lambda \in \mathbb{R}$ $\quad 0 \leq \lambda \leq 4$

$D_1 = D_N$ $\quad D_2 = \dots = D_{N-1}$

0 1 2 3 4

$\lambda_i = 2\left(1 - \cos\frac{\pi i}{N+1}\right)$

$N \to \infty$ $\lambda_1 \to 0$ $\lambda_N \to 4$

end of parenthesis

eigenvalues of A: $\lambda_i = \frac{c^2}{h^2} 2\left(1 - \cos\frac{\pi i}{N+1}\right)$ $i = 1, \dots, N$

$\lambda_N = \frac{c^2}{h^2} 2\left(1 - \cos\frac{N\pi}{N+1}\right)$

$= \frac{c^2}{h^2} 2\left(1 - \cos(\pi - \frac{\pi}{N+1})\right)$

$= \frac{c^2}{h^2} 2\left(1 + \cos\frac{\pi}{N+1}\right)$ $\quad h = \frac{1}{N+1}$

$\cos\frac{\pi}{N+1} = \cos(\pi h) = 1 - \frac{\pi^2 h^2}{2} + O(h^4)$

$\lambda_N = \frac{c^2}{h^2} 2\left(2 - \frac{\pi^2 h^2}{2}\right)$

$= \frac{4c^2}{h^2} + O(1) \xrightarrow[N\to\infty]{h\to 0} \frac{4c^2}{h^2}$

$\zeta^2 \leq \frac{4}{\lambda_N} = \frac{4}{\frac{4c^2}{h^2} + O(1)} \xrightarrow{h\to 0} \frac{h^2}{c^2}$

$\zeta \leq \frac{h}{c}$ $\quad \frac{m}{s}$

s

Num Exp: Set N, do 1000 time steps $n = 1, \dots, M = 10^3$

Increase τ until observe blow up

Computation of $\vec{u}^0$ and $\vec{u}^1$? $\vec{u}^0 = u_0(x_i), i = 1, \dots, N$

$$u(x, \underbrace{\tau}_{t_1}) = u(x,0) + \tau \underbrace{\frac{\partial u}{\partial t}(x,0)}_{v_0(x)} + \frac{\tau^2}{2} \underbrace{\frac{\partial^2 u}{\partial t^2}(x,0)}_{c^2 \frac{\partial^2 u}{\partial x^2}(x,0)\ (\cdot f_0 = 0)} + O(\tau^3)$$

$$u_i^1 = u_i^0 + \tau\, v_0(x_i) + \frac{\tau^2 c^2}{2} \frac{u_{i-1}^0 - 2u_i^0 + u_{i+1}^0}{h^2}$$

Both methods (expl and implicit) converge $O(h^2 + \tau^2)$ under the stab cond

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Extensions:

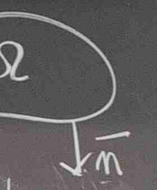
- damping: $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial t} = 0$ mult. by $\frac{\partial u}{\partial t}$ and $\int_0^1 dx$

$$\frac{1}{2} \frac{d}{dt} \left(\int_0^1 \left(\left(\frac{\partial u}{\partial t}\right)^2 + c^2 \left(\frac{\partial u}{\partial x}\right)^2 \right) dx \right) + \alpha \underbrace{\int_0^1 \left(\frac{\partial u}{\partial t}\right)^2 dx}_{\geq 0} = 0$$

Therefore

$$\frac{d}{dt}\underbrace{\int_0^1 \left(\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right) dx}_{\text{decreasing with } t} \leq 0$$

The 2 or 3d wave eq $x \in \Omega \subset \mathbb{R}^d$ $d = 2, 3$



$$\begin{cases} \text{in } \Omega \times (0,T) & \dfrac{\partial^2 u}{\partial t^2} - \underbrace{c^2 \Delta u} = 0 \qquad \Delta u = \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \dfrac{\partial^2 u}{\partial x_3^2} \quad 3d \\ & \text{or } \operatorname{div}\left((c(x,t))^2 \nabla u(x,t)\right) \qquad \vec{x} = (x_1, x_2, x_3) \\ u(x,t) = 0 & x \in \partial\Omega \quad 0 \leq t \leq T \quad \text{boundary conditions} \\ u(x,0) = u_0(x) & x \in \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = v_0(x) & \end{cases}$$

$\zeta^2 \leq \frac{4}{\lambda_N}$ λ_N ?

$A = \frac{c^2}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & \\ & & & & \end{pmatrix}$

$0 \leq \text{eigenv} \leq 4$

first Gerschgorin thm

parenthesis

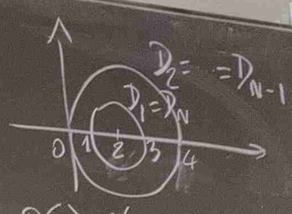
a_{11} ... a_{NN}

$\lambda \in \bigcup_{i=1}^{N} D_i$

D_i = Ball centered at a_{ii} radius $\sum_{\substack{j=1 \\ j\neq i}}^{N} |a_{ij}|$

$A\bar{\varphi} = \lambda\bar{\varphi}$ ... $(a_{ii} - \lambda_i)\varphi_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij}\varphi_j$

$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & \\ & & & & \end{pmatrix}_N$ 1



A sym. $\lambda \in \mathbb{R}$ $0 \leq \lambda \leq 4$

$\lambda_i = 2\left(1 - \cos\frac{\pi i}{N+1}\right)$

$N \to \infty$ $\lambda_1 \to 0$ $\lambda_N \to 4$

end of parenthesis

mult by $\frac{\partial u}{\partial t}$ and $\int_\Omega dx$ to obtain

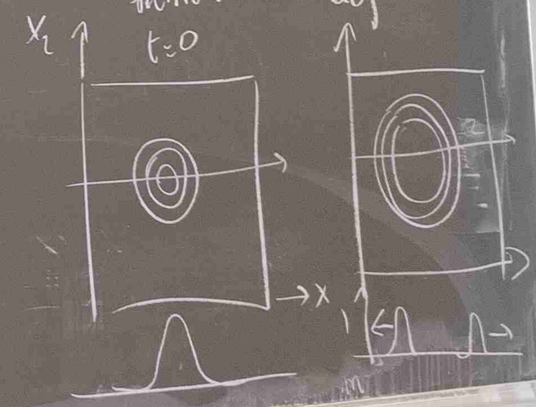
$$\int_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}dx - c^2\int_\Omega \operatorname{div}\nabla u \cdot \frac{\partial u}{\partial t}dx$$

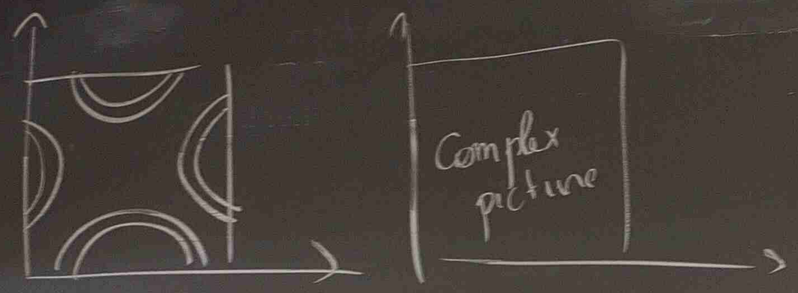
$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx + c^2\int_\Omega \nabla u \nabla\frac{\partial u}{\partial t}dx - c^2\int_{\partial\Omega} \nabla u \cdot n \underbrace{\frac{\partial u}{\partial t}}_{0 \text{ on } \partial\Omega} dx = 0$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\left(\frac{\partial u}{\partial t}\right)^2 + c^2|\nabla u|^2\right)dx = 0$$

kinetic E

def E





$$d=2 \qquad \vec{u}(x_1,x_2,t)=\begin{pmatrix} u_1(\cdot) \\ u_2(\cdot) \end{pmatrix}$$

$$\int \rho\left(\partial_t u_1 + u_1\partial_1 u_1 + u_2\partial_2 u_1\right) - \mu\,\Delta u_1 + \partial_1 p = f_1$$

$$\rho\left(\partial_t u_2 + u_1\partial_1 u_2 + u_2\partial_2 u_2\right) - \mu\,\Delta u_2 + \partial_2 p = f_2$$

$$\partial_1 u_1 + \partial_2 u_2 = 0$$

Euler $\mu = 0$

1753!

→ Existence 1930 Leray

Next time:



Navier-Stokes: $\vec{u}(\vec{x},t) \in \mathbb{R}^d,\ \vec{x} \in \mathbb{R}^d$ $d=2,3$ $0 \le t \le T$ $\quad p(\vec{x},t) \in \mathbb{R}$

$$\int \rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right) - \mu\,\Delta\vec{u} + \vec{\nabla}p = \int \rho\vec{g} \qquad \Omega \times (0,T)$$

$\operatorname{div}\vec{u} = 0$ (liquid)

BC: $u = 0$ $\partial\Omega \times (0,T)$

IC: $u(x,0) = u_0(x)$ $x \in \Omega$

Solving NSeq

Data $\Omega \subset \mathbb{R}^d$ $d=2,3$ $T>0$ $\rho>0$ $\mu>0$ $f: \Omega\times(0,T) \to \mathbb{R}^d$, $(x,t) \to f(x,t)$

$u_0 : \Omega \to \mathbb{R}^d$, $x \to u_0(x)$ s.t.

Unknowns $\bar{u} : \Omega\times(0,T) \to \mathbb{R}^d$ and $p : \Omega\times(0,T) \to \mathbb{R}$

$$\begin{cases} \rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) - \mu\Delta u + \nabla p = f & \Omega\times(0,T) \\ \operatorname{div} u = 0 & \Omega\times(0,T) \\ u = 0 & \partial\Omega\times(0,T) \\ u(\cdot,0) = u_0 & \text{in } \Omega \end{cases}$$

Euler 1753 ($\mu=0$)

Navier (1822) Stokes (1853) $\mu>0$

Leray (1934)

$d=2$ $\bar{u}(x_1,x_2,t) = \begin{pmatrix} u_1(x_1,x_2,t) \\ u_2(x_1,x_2,t) \end{pmatrix}$ $\bar{u}\cdot\bar{\nabla} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} = u_1\partial_1 + u_2\partial_2$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_1\partial_1 u_i + u_2\partial_2 u_i\right) - \mu\Delta u_i + \partial_i p = f_i \quad i=1,2$$

A priori estimate: $\frac{1}{2}\frac{d}{dt}\rho\int_\Omega \bar{u}\cdot\bar{u}\,dx + \mu\int_\Omega |\nabla\bar{u}|^2 dx = \int_\Omega \bar{f}\cdot\bar{u}\,dx$

Acoustics:

$\operatorname{div} \vec{u} \longrightarrow \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{u})$

linearize around $\rho_0 = cst$ and $\vec{u}_0 = \vec{0}$, $p = c^2 \rho$

$$\begin{cases} \frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} u = 0 \\ \rho_0 \frac{\partial u}{\partial t} + c^2 \nabla \rho = 0 \end{cases}$$

$$\frac{\partial^2 \rho}{\partial t^2} + \rho_0 \operatorname{div} \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial^2 \rho}{\partial t^2} - c^2 \operatorname{div} \nabla \rho = 0 \quad \text{wave eq for } \rho$$

Stokes:

$\frac{\partial}{\partial t} = 0$ neglect $(\vec{u} \cdot \nabla)\vec{u}$

$$\begin{cases} -\mu \Delta u + \nabla p = f & \Omega \\ \operatorname{div} u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

p is given up to a cst (if p is set then $p + cst$ is set)

After space discretization (Finite Diff, Finite Element, Finite Volumes)

Linear system:

(1) $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{f} \\ \vec{0} \end{pmatrix}$

$$\begin{cases} A\vec{u} + B^T p = \vec{f} \\ B\vec{u} = \vec{0} \end{cases}$$

$A: -\mu \Delta \vec{u}$ $\quad B: \operatorname{div} \vec{u}$

$B^T: \vec{\nabla} p$

$\vec{f}: f$

Matrix invertible?

$\forall f \; \exists! (\vec{u}, \vec{p})$ s.t. (1) is satisfied?

Ker?

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix} \; ? \Rightarrow (\vec{u} = 0 \; \vec{p} = 0)$$

$$\begin{cases} A\vec{u} + B^T \vec{p} = 0 \\ B\vec{u} = 0 \end{cases}$$

A square matrix $A \in \mathbb{R}^{m \times m}$

B rectangular $B \in \mathbb{R}^{m \times m}$

$\vec{u} \in \mathbb{R}^m$ $\quad \vec{p} \in \mathbb{R}^m$

Scalar prod with $\vec{u}, \vec{p}$

$$\left(\begin{array}{rl} \vec{u}^T A \vec{u} + \vec{u}^T B^T \vec{p} &= 0 \\ - \left(\vec{p}^T B u \right. &= 0 \end{array}\right)$$

$\vec{u}^T A \vec{u} = 0 \Rightarrow \vec{u} = 0$

A is sym. pos. def.

Then $B^T \vec{p} = -A\vec{u} = \vec{0} \Rightarrow \vec{p} = \vec{0}$

$\operatorname{Ker} B^T = 0$

If A is s.p.d and $\operatorname{Ker} B^T = 0$

then $\forall \vec{f} \in \mathbb{R}^m$ there exists a unique $\vec{u}, \vec{p}$ s.t.

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{f} \\ \vec{g} \end{pmatrix}$$

Remark: Finding $\vec{u}, \vec{p}$ corresponds to a minimization pbm with eq. constraints

$$\min_{\substack{\vec{v} \in \mathbb{R}^n \\ \text{constraint} \\ B\vec{v} = \vec{0}}} \left(\frac{1}{2} \vec{v}^T A \vec{v} - \vec{f}^T \vec{v} \right) \qquad (2)$$

Lagrangian: $\mathcal{L}(\vec{v}, \vec{q}) = \frac{1}{2} \vec{v}^T A \vec{v} - \vec{f}^T \vec{v} + \vec{q}^T B \vec{v}$

$\partial_v \mathcal{L}(v, q) = A\vec{v} - \vec{f} + B^T \vec{q}$

$\partial_q \mathcal{L}(v, q) = B\vec{v}$

solution of (2): $\vec{u}, \vec{p}$ s.t.

$\partial_v \mathcal{L}(u, p) = 0$

$\partial_q \mathcal{L}(u, p) = 0$

Issues: $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} \vec{u} \\ \vec{p} \end{pmatrix}$ big linear system $u_1 u_2 u_3 p$ at each vertex of the grid, $\vec{u}$ and p are coupled

$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$ is not sym. pos. def. cannot use Conj. grad but GMRES

idea: eliminate $\vec{u}$

$$\begin{cases} A\vec{u} + B^T p = \vec{f} \\ B\vec{u} = \vec{0} \end{cases} \quad \begin{cases} \vec{u} = A^{-1}\left(\vec{f} - B^T\vec{p}\right) \\ BA^{-1}B^T\vec{p} = BA^{-1}\vec{f} \end{cases}$$

$BA^{-1}B^T$ is s.p.d: $\forall \vec{q} \in \mathbb{R}^m$

$$\underbrace{\vec{q}^T B A^{-1} B^T}_{\vec{v}^T} \underbrace{\vec{q}}_{\vec{v}} = \vec{v}^T A^{-1}\vec{v}$$

A is spd: $\lambda_1 \|v\|^2 \le \vec{v}^T A \vec{v} \le \lambda_n \|v\|^2$ $\left(A\vec{\Psi}_i = \lambda_i \vec{\Psi}_i \quad \vec{\Psi}_i^T \vec{\Psi}_j = \delta_{ij} \quad 0 < \lambda_1 \le \dots \le \lambda_n\right)$

thus $\frac{1}{\lambda_n}\|v\|^2 \le q^T B A^{-1} B^T \vec{q} \le \frac{1}{\lambda_1}\|v\|^2$

Since $\vec{v} = B^T \vec{q}$

$$\frac{1}{\lambda_m} C^2 \|q\|^2 \leq \frac{1}{\lambda_m} \|B^T \vec{q}\|^2 \leq \vec{q}^T B A^{-1} B^T \vec{q} \leq \frac{1}{\lambda_1} \|B^T \vec{q}\|^2 \leq \frac{1}{\lambda_1} \|B^T\| \, \|\vec{q}\|^2$$

if $\exists C > 0 \quad \forall \vec{q} \in \mathbb{R}^m \quad \|B^T \vec{q}\| \geq C \|q\|$

thus $BA^{-1}B^T$ is s.p.d. and we have an estimate of its eigenvalues

We can apply conjugate gradient to $BA^{-1}B^T p = BA^{-1} f$:

p^0

$n = 0, 1, 2, \ldots$

Compute $BA^{-1}B^T p^n$

$BA^{-1}B^T p^n = Bz^n$ with $Az^n = B^T p^n$

($BA^{-1}B^T p^n$: z^n)

a linear system $Az =$ something has to be solved at each iter.

Another method (less efficient than CG)

Uzawa (exercice) $\alpha > 0$

p^0

$n = 0, 1, \ldots$

Find $\vec{u}^n$ s.t. $A\vec{u}^n = f - B^T \vec{p}^n$

Update p: $\vec{p}^{n+1} = \vec{p}^n + \alpha B \vec{u}^n$

Can be proved to converge ($p^n \xrightarrow{n \to \infty} p$) provided a condition on B^T holds and if α is suff. small

Continuous version of Uzawa:

$$\begin{cases} -\mu \Delta u + \nabla p = f & \times u \quad \int_\Omega dx \\ \operatorname{div} u + \alpha \frac{\partial p}{\partial t} = 0 & \times p \quad \int_\Omega dx \end{cases}$$

A priori estimate

$$\int_\Omega \mu \|\nabla u\|^2 - \int_\Omega p \operatorname{div} u + \int_\Omega \operatorname{div} u \, p + \alpha \int_\Omega \frac{\partial p}{\partial t} p = \int_\Omega f u \, dx$$

$$\int_\Omega \mu \|\nabla u\|^2 + \alpha \frac{1}{2} \frac{d}{dt} \int_\Omega p^2 \, dx = 0$$

Euler: (simplified)

$$\begin{cases} \rho \frac{\partial u}{\partial t} + \nabla p = f & \Omega \\ \operatorname{div} u = 0 & \Omega \end{cases}$$

$$\frac{\partial}{\partial t} \operatorname{div} u + \operatorname{div} \nabla p = \operatorname{div} f$$

$$\begin{cases} -\Delta p = -\operatorname{div} f & \Omega \\ \nabla p \cdot \vec{n} = f \cdot n - \rho \frac{\partial^2}{\partial t} u \cdot n & \partial\Omega \end{cases}$$

Navier-Stokes

After space discret, we obtain a differential system.

$$\begin{cases} \rho \vec{u}'(t) - \vec{C}(\vec{u}(t)) \\ + A\vec{u}(t) + B^T \vec{p}(t) = \vec{f}(t) \\ B\vec{u}(t) = \vec{0} \end{cases}$$

Implicit Euler

$$\begin{cases} \rho \frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} - \vec{C}(\vec{u}^{n+1}) + A\vec{u}^{n+1} + B^T \vec{p}^{n+1} = \vec{f}(t_{n+1}) \\ B\vec{u}^{n+1} = \vec{0} \end{cases}$$

Nonlinear System (big, couples u and p)

Splitting scheme!

prediction: $\frac{\vec{u}^{n+1/2} - \vec{u}^{n}}{\tau} + C(\vec{u}^{n+1/2}) + A\vec{u}^{n+1/2} = f(t_{n+1/2})$ (3)

$$\left(\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) - \mu\Delta u = f \right)$$

Correction:
$$\begin{cases} \frac{u^{n+1} - u^{n+1/2}}{\tau} + B^T p^{n+1} = 0 & (4) \\ Bu^{n+1} = 0 & (5) \end{cases}$$

Apply B to (4):

$$\frac{\overbrace{Bu^{n+1}}^{0} - Bu^{n+1/2}}{\tau} + BB^T p^{n+1} = 0$$

$$BB^T p^{n+1} = -\frac{1}{\tau} B\vec{u}^{n+1/2} \quad (6)$$

Given $\vec{u}^n$
Compute $u^{n+1/2}$ with (3)
p^{n+1} with (6)
u^{n+1} with (4)

- ODE RK schemes → 1Q
- Elliptic → 1Q or 2Q
- Parab → 1Q
- Hyperb (transport or wave) → 1Q
- $\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$ $t_0 \leq t \leq T$, $\vec{y_0} \in \mathbb{R}^d$

Euler scheme, explicit or implicit

$d = 1$

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n)$$

$$\frac{y(t_{n+1}) - y(t_n)}{h} = f(t_n, y(t_n)) + r_n \quad |r_n| \leq Ch$$

$$E_{n+1} = E_n + h\left(f(t_n, y(t_n)) - f(t_n, y_n)\right) + h r_n$$

$$|E_{n+1}| \leq |E_n|(1 + hL) + Ch^2$$

$$|E_N| \le |E_0| (1+hL)^N + Ch^2 \frac{(1+hL)^N - 1}{hL} \le \frac{C}{L}\left(e^{\frac{RLN}{T}} - 1\right) R$$

$t_N = Nh = T$

$$1 + (1+hL) + (1+hL)^2 + \dots + (1+hL)^{N-1} = \frac{(1+hL)^N - 1}{1+hL-1}$$

$$e^x = 1 + x + \frac{x^2}{2} e^s \qquad 0 < s < x$$

$$\geq 1 + x$$

$y' = \lambda y \quad \lambda < 0$

$y_{n+1} = y_n (1 + \lambda h)$

$y_N = y_0 (1+\lambda h)^N$

$y(t_N) = y_0 e^{\lambda N h}$

$y(t_N) - y_N = y_0 \left(e^{\lambda N h} - (1+\lambda h)^N\right)$

$= y_0 \left(e^{\lambda h} - (1+\lambda h)\right)\left(e^{\lambda(N-1)h} + \dots + (1+\lambda h)^{N-1}\right)$

$\le N \left|e^{\lambda h} - (1+\lambda h)\right| \max |1+\lambda h|^k$

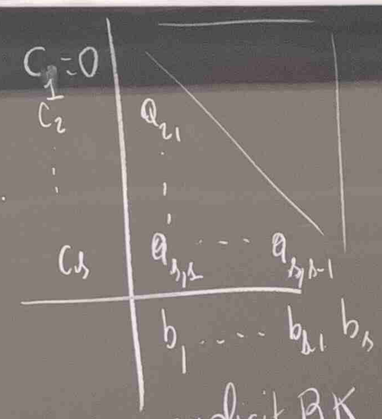
$h < \frac{2}{|\lambda|}$

$y' + Ay = 0 \qquad A$ sym. pos. def.

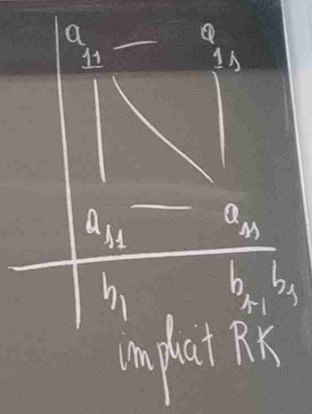
RK expl. s stages Y_1 approx $y(\underbrace{t+h}_{t_1})$

$$k_1 = f(t_0, y_0)$$
$$k_2 = f(t_0 + c_2 h, y_0 + h a_{21} k_1$$
$$\vdots$$
$$k_s = f(t_0 + c_s h, y_0 + h(a_{s1} k_1 + \dots + a_{s,s-1} k_s)$$

$$Y_1 = y_0 + h(b_1 k_1 + \dots + b_s k_s)$$



explicit RK



implicit RK

c_i, a_{ij}, b_i $\quad y(t_1) - y_1 = O(h^{p+1})$?

$p=1 \quad \sum_{i=1}^{s} b_i = 1$

$p=2 \quad \sum_{i,j=1}^{s} b_i a_{ij} = 1/2$

$p=3$

$p=4$

impl. or expl.

Proof: $y'(t) = f(y(t))$

$c_i = \sum_{j=1}^{s} a_{ij}$

$y(t_0+h) = \underbrace{y(t_0)}_{y_0} + h \underbrace{y'(t_0)}_{f(y_0)} + O(h^2)$

$y_1(h) = y_0 + \underbrace{h \sum_{i=1}^{s} b_i \, k_i(h)}_{\phi(h)}$

$k_i(h) = f\left(y_0 + h \sum_{j=1}^{s} a_{ij} k_j(h)\right)$

$y_1(h) = y_0 + \underbrace{\phi(0)}_{0} + h\phi'(0) + O(h^2)$

$\phi'(h) = \sum_{i=1}^{s} b_i k_i(h) + h \sum_{i=1}^{s} b_i k_i'(h)$

$\phi'(0) = \sum_{i=1}^{s} b_i \underbrace{k_i(0)}_{f(y_0)}$

$y(t_0+h) - y_1(h) = h\left(\underbrace{1 - \sum_{i=1}^{s} b_i}_{\text{if } 0,\ p=1}\right) f(y_0) + O(h^2)$

$y' = f(y)$

$y'' = f'(y)y'$

$y''(t_0) = \underbrace{f'(y_0)}_{\text{matrix } d\times d} \underbrace{f(y_0)}_{\text{vector } d}$

convergence $|y(t_N) - y_N|$? expl. RK

$y_{n+1} = y_n + h_n \phi(t_n, y_n, h_n)$

$|y(t+h) - (y(t) + h\ \phi(t, y(t), h))| \leq C h^{p+1}$ $\forall t, h$

$|\phi(t, z_1, h) - \phi(t, z_2, h)| \leq \Lambda |z_1 - z_2|$ $\forall t, h$ $\forall z_1, z_2$

$|y(t_N) - y_N| \leq \frac{C}{\Lambda} \left(e^{L(t_N - t_0)} - 1\right) h_{max}^p$

$h_{max} = \max(h_{N-1}, \dots, h_0)$

$|y(t_{n+1}) - y_{n+1}| = |y(t_{n+1}) - (y_n + h_n \phi(t_n, y_n, h_n))|$

add and remove $y(t_n) + h_n \phi(t_n, y(t_n), h_n)$

i approx $u(x_i)$ $i = 1, \dots, N$

$x_i = ih$ $h = \frac{1}{N+1}$

h^2

$A\vec{u} = \vec{f}$

$A\vec{U} = \vec{f} + \vec{r}$ $\vec{U} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ $\vec{u} = \begin{pmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix}$

$C = \frac{1}{12} \max_{0 \leq x \leq 1} \left|\frac{d^4 u}{dx^4}\right|$

$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$

$A(\vec{U}-\vec{u})=\vec{r}$

lemma: $A\vec{v}=\vec{g}$ $\quad \|\vec{v}\|_\infty \le \frac{1}{8}\|\vec{g}\|_\infty = \frac{1}{8}\max_{1\le i\le N}|g_i|$

$$\|\vec{U}-\vec{u}\|_\infty \le \frac{1}{8}\,\frac{1}{12}\max_{0\le x\le 1}|u^{(4)}(x)|\,h^2$$

$$\|\vec{v}\|_2^2 = \sum_{i=1}^{N} v_i^2$$

$A\vec{\varphi}_i = \lambda_i \vec{\varphi}_i \qquad \vec{\varphi}_i^T\vec{\varphi}_j = \delta_{ij} \qquad 0<\lambda_1\le\dots\le\lambda_N \qquad \lambda_i = 2\left(1-\cos\frac{i\pi}{N+1}\right)(N+1)^2$

$\lambda_1 \sim \pi^2 \qquad \left(-\varphi''(x)=\lambda\varphi(x),\ \varphi(0)=\varphi(1)=0 \quad \varphi_i(x)=\sin(i\pi x),\ \lambda_i=\pi^2 i^2\right)$

$$\cos x = 1-\frac{x^2}{2}+O(x^4)$$

$$1-\cos x = \frac{x^2}{2}+O(x^4)$$

$$(N+1)^2 2\left(1-\cos\frac{\pi}{N+1}\right) = \frac{2\pi^2}{2}+O\left(\frac{1}{N^2}\right)$$

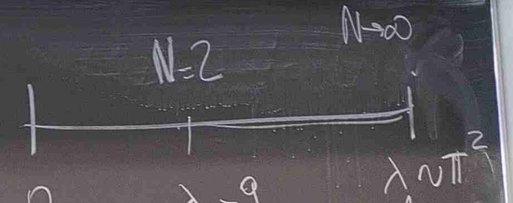
$N=2$

$$\begin{pmatrix}2&-1\\-1&2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=1\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}2&-1\\-1&2\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=3\begin{pmatrix}1\\-1\end{pmatrix}$$

$\times 9$

$N=2$ $\qquad N\to\infty$



$$\lambda_1\|\vec{v}\|^2 \le \vec{v}^T A\vec{v} \le \lambda_N\|\vec{v}\|^2 \qquad \forall \vec{v}\in\mathbb{R}^N$$

$$9\,\|\vec{U}-\vec{u}\|_2^2 \le (\vec{U}-\vec{u})^T A(\vec{U}-\vec{u}) = \vec{r}^T(\vec{U}-\vec{u}) \le \|\vec{r}\|_2\,\|\vec{U}-\vec{u}\|_2$$

$\|\vec{u}-\vec{u}_h\|_2 \le \frac{1}{9}\|\vec{r}\|_2$ $\quad |r_i| \le Ch^2$

$\|\vec{r}\|_2 \le Ch^2\sqrt{N} \le Ch^{3/2}$

$w_i = x_i\frac{(1-x_i)}{2}$ $\quad \begin{pmatrix} -w''(x)=1 \\ w(0)=0 \\ w(1)=0 \end{pmatrix}$ $\quad w(x)=\frac{x(1-x)}{2}$

Proof lemma: $\circ\ A\vec{w} \ge \vec{1}$ $\quad (=\vec{1})$

$\circ\ (A\vec{z} \ge \vec{0}) \Rightarrow (\vec{z} \ge \vec{0})$

$A(\vec{w}\|\vec{g}\|_\infty - \vec{v}) \ge \vec{1}\|\vec{g}\|_\infty - \vec{g} \ge 0$

$A(\vec{w}\|\vec{g}\|_\infty + \vec{v}) \ge \vec{1}\|\vec{g}\|_\infty + \vec{g} \ge 0$

$-\vec{w}\|\vec{g}\|_\infty \le \vec{v} \le \vec{w}\|\vec{g}\|_\infty$

$|g_i| \le \|\vec{g}\|_\infty$

$-\|g\|_\infty \le g_i \le \|g\|_\infty$

$\|\vec{v}\|_\infty \le \|\vec{w}\|_\infty \|\vec{g}\|_\infty$

2D: $\begin{cases} -\Delta u(x_1,x_2) = f(x_1,x_2) & (x_1,x_2) \in \underbrace{]0,1[^2}_{\Omega} \\ u(x_1,x_2) = 0 & \text{on } \partial\Omega \end{cases}$

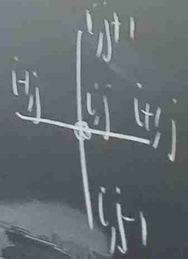


$L^2 = N$

$1 \le i,j \le L$

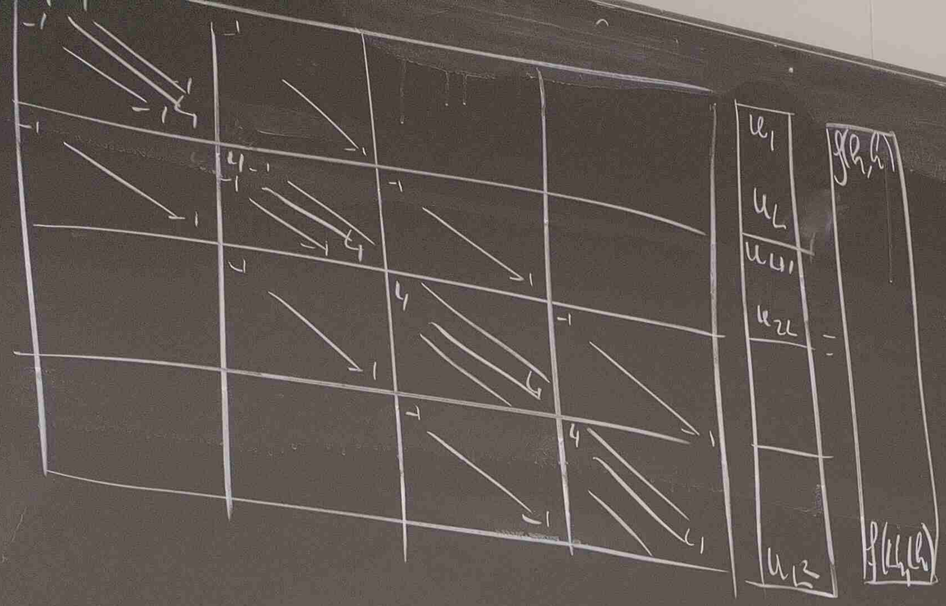
$h = \frac{1}{L+1}$

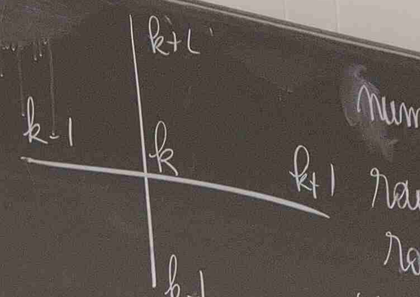
$U_{i,j} \simeq u(ih, jh)$



$$\frac{-U_{i,j-1} - U_{i-1,j} + 4U_{i,j} - U_{i+1,j} - U_{i,j+1}}{h^2} = f(ih, jh)$$

$\frac{1}{h^2}$



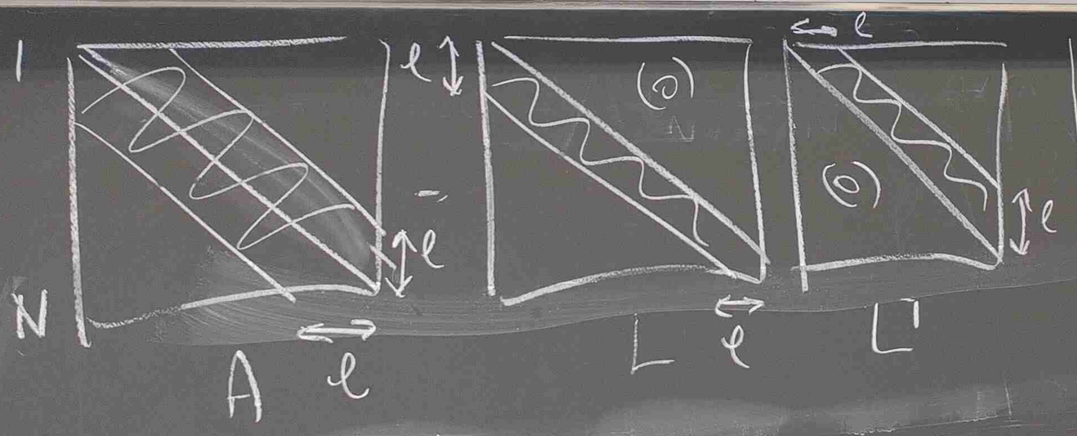


numbering row by row

$U_{ij} \to u_k$

$$A = \frac{1}{h^2}\left(I_L \otimes D_L + D_L \otimes I_L\right)$$

$$D_L = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}_L$$



memory : $O(Ne)$

CPU : $O(Ne^2)$

	Chol	CG
Mem	$O(L^3)$	$O(L^2)$
CPU	$O(L^2 L^2)$	$O(L^2 L)$

$$\vec{u}^{n+1} = (I - \tau A)\vec{u}^n + \tau \vec{g}^n$$

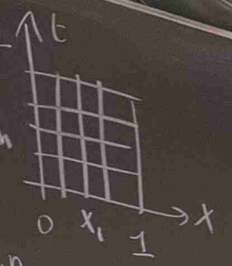
$$\|\vec{u}^{n+1}\|_2 \leq \max_{1 \leq i \leq n} |1 - \tau\mu_i| \, \|\vec{u}^n\|_2 + \tau \|\vec{g}^n\|_2$$

$$\|B\vec{x}\|_2 \leq \max_{1 \leq i \leq N} |\mu_i| \, \|\vec{x}\|_2$$

$$-1 \leq 1 - \tau\mu_i \leq 1$$

$$\tau \leq \frac{2}{\lambda_N} \sim \frac{h^2}{2}$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f$$



$$u(x_i, t_m) \simeq u_i^m$$

$$\frac{u_i^{m+1} - u_i^m}{\tau} - \frac{u_{i-1}^m - 2u_i^m + u_{i+1}^m}{h^2} = f(x_i, t_m)$$

$$\frac{u(x_i, t_{m+1}) - u(x_i, t_m)}{\tau} + \frac{u(x_{i-1}, t_m) - 2u(x_i, t_m) + u(x_{i+1}, t_m)}{h^2} = f(x_i, t_m) + \eta_i^m \quad |\eta_i^m| \le C(h^2 + \tau)$$

$$\|\bar{u}^{m+1}\|_\infty \le \|\bar{u}^m\|_\infty + \tau \|\bar{f}^m\|_\infty \qquad \text{if } \tau \le \frac{h^2}{2}$$

$$k \quad |u_k^{m+1}| = \|\bar{u}^{m+1}\|_\infty \ge |u_i^{m+1}| \qquad i = 1, \dots, N$$

$$|u_k^{m+1}| = |u_k^m| \left(1 - \frac{2\tau}{h^2}\right) + \left(u_{k-1}^m + u_{k+1}^m\right)\frac{\tau}{h^2} + \tau |f(x_i, t_m)|$$

$$\le \left(1 - \frac{2\tau}{h^2}\right) |u_k^m| + \frac{\tau}{h^2}\left(|u_{k-1}^m| + |u_{k+1}^m|\right) + \tau |f(x_i, t_m)|$$

$$\|\bar{u}^{m+1}\|_\infty \le \left(1 - \frac{2\tau}{h^2} + \frac{2\tau}{h^2}\right) \|\bar{u}^m\|_\infty + \tau \|\bar{f}^m\|_\infty$$