EPFL - Marco Picasso - Fall 2025

Advanced Numerical Analysis Exam

January 24, 2025

Question 1

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a smooth given function, $t_0, T \in \mathbb{R}, t_0 < T, y_0 \in \mathbb{R}$. Let y be the solution of

$$\begin{cases} y'(t) = f(t, y(t)), \quad t_0 < t \leqslant T, \\ y(t_0) = y_0. \end{cases}$$

a) Given $h > 0$ write a s stages explicit Runge-Kutta method with coefficients c_i, a_{ij}, b_i to compute y_1, an approximation of $y(t_0 + h)$.

b) In the case when $s = 4$, $c_1 = 0$, $c_i = \sum_{j=1}^{s} a_{ij}, i = 1, \ldots, s$, the order 4 conditions write

$$\begin{cases} b_1 + b_2 + b_3 + b_4 = 1 \\ b_2c_2 + b_3c_3 + b_4c_4 = \frac{1}{2} \\ b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = \frac{1}{3} \\ b_3a_{32}c_2 + b_4(a_{42}c_2 + a_{43}c_3) = \frac{1}{6} \\ b_1c_1^3 + b_2c_2^3 + b_3c_3^3 = \frac{1}{4} \\ b_3c_3a_{32}c_2 + b_4c_4(a_{42}c_2 + a_{43}c_3) = \frac{1}{8} \\ b_3a_{32}c_2^2 + b_4(a_{42}c_2^2 + a_{43}c_3^2) = \frac{1}{12} \\ b_4a_{43}a_{32}c_2 = \frac{1}{24} \end{cases}$$

Let N be a positive integer, let $h = \frac{T}{N}$ be the time step and $t_n = nh, n = 0, 1, ..., N$. Let

$$p_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

In the case when $t_0 = 0, n = 1$ and $f(t, y) = \lambda y, \lambda < 0$, prove that, if the order 4 conditions are satisfied, then

$$y_1 = p_4(\lambda h)y_0.$$

Prove that $y(t_n) - y_n = y_0(e^{h\lambda n} - p_4(\lambda h)^n)$ and that $\exists C > 0, \forall T > 0, \forall \lambda < 0, \forall h > 0$ such that $|p_4(\lambda h)| \leqslant 1$ we have

$$|y(t_N) - y_N| \leqslant CT|\lambda^5|h^4.$$

c) Give an example of an explicit Runge Kutta method of order 2 with 2 stages.
Propose an adaptive algorithm that uses an order 1 and an order 2 explicit Runge Kutta method so that the error at each time step is less than a preset tolerance TOL.

Question 2

Which problem is approximated by the matlab/octave file exam1.m?

```
function [u]=exam1(N,M)

h = 1/(N+1)
tau=1/M
I=speye(N,N);diag = 2*I;subd=-sparse(2:N,1:N-1,1,N,N);
A=(N+1)^2*(diag+subd+subd');
u0=sparse(N,1);
u1=sparse(N,1);
for i=1:N
```

```
  u0(i)=sin(pi*i*h);
end
u1=(I-tau*tau/2*A)*u0;
mat=2*I-tau*tau*A;
err=sparse(N,1);
for n=1:M
 u2=mat*u1-u0;
 u0=u1;
 u1=u2;
end
for i=1:N
   err(i)=u2(i)-sin(pi*i*h)*cos(pi*n*tau);
end
fprintf('err inf=%e \n',norm(err,inf))
fprintf('err 2=%e \n',norm(err,2))
end
```

a. Write the problem to be approached and the numerical scheme.

b. Discuss the numerical results detailed hereafter:

```
octave:13> exam1(79,80)
h = 0.012500
tau = 0.012500
err inf=7.709638e-04
err 2=4.876003e-03

octave:14> exam1(159,160)
h = 6.2500e-03
tau = 6.2500e-03
err inf=1.927595e-04
err 2=1.724094e-03
```

Question 3

Let $u : [0,1] \to \mathbb{R}$ be such that,

$$\begin{cases} -u''(x) + u'(x) = 1, & 0 < x < 1, \\ u(0) = 0, & \\ u(1) = 0. & \end{cases} \tag{1}$$

Let $N \geqslant 2$, $h = \frac{1}{N+1}$, $x_i = ih$, $i = 0, 1, ..., N+1$.

a. Write a centered finite difference scheme to compute u_i approximations of $u(x_i)$, $i = 1, ..., N$.

b. Write the scheme as

$$(A + B)\vec{u} = \vec{1},$$

where A is symmetric positive definite and $B = -B^T$.

c. Prove that

$$(A + B)\vec{U} = \vec{1} + \vec{r},$$

where $\vec{U}$ has components $u(x_i)$ and $|r_i| \leqslant Ch^2$ (make explicit what is C and the assumption on u).

d. Let λ_1 be the smallest eigenvalue of A. Prove that $||\vec{U} - \vec{u}||_2 \leqslant \frac{Ch^{3/2}}{\lambda_1}$.

Question 4

Let $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) + u(x,t) = 1, & 0 < x < 1,\ 0 < t \leqslant T, \\ u(0,t) = 0,\ u(1,t) = 0, & 0 < t \leqslant T, \\ u(x,0) = sin(\pi x), & 0 < x < 1. \end{cases}$$

Let $N,\ M$ be two positive integers, $h = \frac{1}{N+1}$, $\tau = \frac{T}{M}$, $x_i = ih$, $i = 0, ..., N+1$, $t_n = n\tau$, $n = 0, ..., M$. Consider the scheme for $n = 0, ..., M-1$

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\tau} + A\vec{u}^{n+1} + \vec{u}^{n+1} = \vec{1}$$

where $\vec{u}^n$ has components u_i^n and

$$A = (N+1)^2 \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & (0) \\ (0) & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

a. Prove that for all $N,\ M$,

$$||\vec{u}^{n+1}||_\infty \leqslant ||\vec{u}^n||_\infty + \tau.$$

where $||\vec{u}^n||_\infty = \max\limits_{1 \leqslant i \leqslant N} |u_i^n|$.

b. Prove that, if u is C^4 in space and C^2 in time,

$$\frac{\vec{U}^{n+1} - \vec{U}^n}{\tau} + A\vec{U}^{n+1} + \vec{U}^{n+1} = \vec{1} + \vec{r}^{n+1}$$

where $\vec{U}^n \in \mathbb{R}^N$ has components $u(x_i, t_n)$ and $|r_i^{n+1}| \leqslant C(h^2 + \tau)$ (C has to be made explicit).

c. Prove that

$$||\vec{U}^{n+1} - \vec{u}^{n+1}||_\infty \leqslant ||\vec{U}^n - \vec{u}^n||_\infty + \tau||\vec{r}^{n+1}||_\infty,$$

and that

$$||\vec{U}^M - \vec{u}^M||_\infty \leqslant CT(h^2 + \tau).$$

Question 5

Given $f : [0,1] \times [0,T] \to \mathbb{R}$ and $u_0 : [0,1] \times [0,T] \to \mathbb{R}$ continuous, we are searching for $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial u}{\partial x}(x,t) = f(x,t), & 0 < x < 1,\ 0 < t \leqslant T, \\ u(1,t) = 0, & 0 < t \leqslant T, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases}$$

Let N, M be two positive integers, $h = \frac{1}{N+1}$, $\tau = \frac{T}{M}$, $x_i = ih$, $i = 0, 1, \dots, N+1$, $t_n = n\tau$, $n = 0, 1, \dots, M$. The following scheme is considered (u_i^n is an approximation of $u(x_i, t_n)$):

$$\frac{u_i^{n+1} - u_i^n}{\tau} - \frac{u_{i+1}^n - u_i^n}{h} = f(x_i, t_n).$$

Let $\vec{u}^n \in \mathbb{R}^N$ be the vector of components $u(x_i, t_n)$, $\|\vec{u}^n\|_\infty = \max_{1 \leqslant i \leqslant N} |u_i^n|$.
Let $\vec{f}^n \in \mathbb{R}^N$ be the vector of components $f(x_i, t_n)$.
Assume that $\tau \leqslant h$, $u \in C^2([0,1] \times [0,T])$ and prove that:

- $$\|\vec{u}^{n+1}\|_\infty \leqslant \|\vec{u}^n\|_\infty + \tau\|\vec{f}^n\|_\infty$$

- $$\frac{u_i(x_i, t_{n+1}) - u_i(x_i, t_n)}{\tau} - \frac{u(x_{i+1}, t_n) - u(x_i, t_n)}{h} = f(x_i, t_n) + r_i^n$$

  with $\|\vec{r}^n\| < C(h + \tau)$ where C must be made explicit

- $$\|\vec{U}^M - \vec{u}^M\|_\infty < CT(h + \tau)$$